HANDBOOK OF ELECTRONIC DESIGN AND ANALYSIS PROCEDURES USING PROGRAMMABLE CALCULATORS

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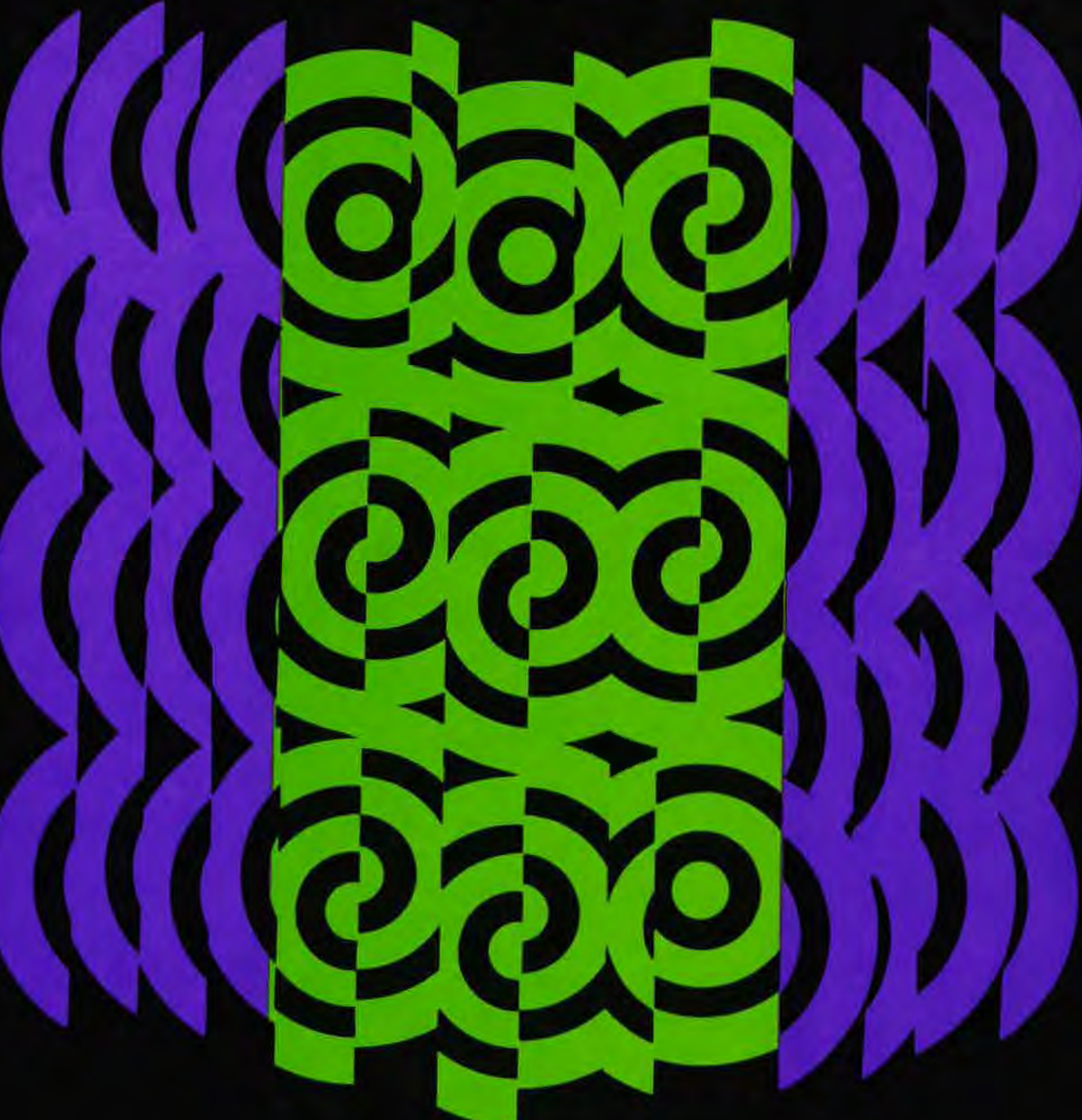

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INTRODUCTION

This book provides programming techniques and programs to make the fully programmable calculator a valuable design tool for the working engineer. This book is not specifically intended to be a textbook on calculator programming, although documented programs can serve this purpose. Three books can be recommended for programming methods and algorithms: Jon M. Smith, "Scientific Analysis on the Pocket Calculator," Wiley 1975, John Ball, "Algorithms for RPN Calculators," Wiley 1978, and Richard W. Hamming, "Numerical Methods for Scientists and Engineers," McGraw-Hill 1973.

Many programs in this book are meant to be used in sets, i.e., the output of one program becomes the input for another through common storage register allocations. The description of each program is meant to stand alone, and consists of the following parts:

1) Problem description with pertinent equations,
2) Program operating instructions,
3) One or more program examples,
4) Equation and method derivation, or references,
5) Annotated program listing, which is its own flowchart.

Part 4 is not present in every program.

This program ordering was chosen so the variable definitions and operating instructions are immediately available to the experienced user. Should the user wish more information or background on the program and equations, or the details of the program operation, this material is also available, but is placed after the operating instructions.

Although the program language, and resulting program flow is tailored to the Hewlett-Packard (HP) fully programmable calculators, the HP-67/97, the annotated program listing/flowchart can be used as a basis for generating programs in other languages.

The language of the Texas Instruments (TI) fully programmable calculator, the TI-59, is not very different from the HP-67/97 language when considered on a gross scale, therefore, the HP-67/97 programs may be easily translated into the language of the TI-59. While it is easy to write a program from equations and flowcharts, the new program must still be debugged. Translating a program that has already been tested and debugged can lead to a new program that has no bugs at all. The TI program translation will also closely follow the flow of the original HP program.

The differences between the HP and TI languages are mainly in format and not in form. Because the TI-59 has few merged keycodes, must use parentheses to set hierarchy, and must branch to a label or line number as the result of a true conditional test, the TI-59 program will be longer than the mating HP-67/97 program. This increased program length is generally not a detriment as it is accommodated by the larger program memory available in the TI-59. Because the TI-59 always starts label searches from the top of the program, the program execution time can also be longer unless direct addressing is used, or the labeled subroutines are placed at the beginning of the program.

Since the TI-59 does not have a stack to hold the results of an equals operation, a set of scratchpad registers must be set aside to hold those intermediate results normally retained in the HP-67/97 stack. Results residing in the TI-59 display register after the equals operation will permanently disappear unless stored before subsequent operations are performed.

The arithmetic hierarchy of the Algebraic Operating System (AOS) can sometimes be a problem which becomes particularly apparent when calling subroutines. If an equals operation does not precede the subroutine call, the subroutine hierarchy will be dependent on the hierarchy set up in the main program. To make the subroutine hierarchy independent of the main program, the subroutine should always start with an open parenthesis and terminate with a close parenthesis. This rule can be extended to the "go to" command also. The last statement prior to the unconditional jump should be an equals to terminate all pending operations. It will cause no harm to have an open parenthesis as the first statement after the label that is the jump destination. The TI-59 has enough program memory so that, whenever in doubt, parenthesis can be inserted to establish unconditional arithmetic hierarchy.

The TI-59 does not have the equivalent of the HP-67/97 flag 3 function where flag 3 is automatically set whenever numeric data

is entered from the keyboard. Because of this difference, the convenience feature existing in most of the HP-67/97 programs herein where the execution of a user definable key such as "A" without numeric entry results in the currently stored parameter value being displayed cannot be translated to the TI-59 program.

None of the TI-59 flags are test cleared, while flags 2 and 3 of the HP-67/97 are test cleared, thus, clear flag statements may be required in the TI-59 program and subroutines involving the use of flags 2 and 3.

The HP-67/97 and the TI-59 both have user definable labels A through E, and a through e (the latter are designated A' through E' on the TI-59). Executing these keys from the keyboard acts like a subroutine call on either machine: the program pointer jumps to the designated label, and program execution begins. The HP-67/97 and the TI-59 are different in the labels called "common labels" by TI, i.e., labels other than the user definable ones. HP uses the label designators 0 through 9, and a given label may be used more than once as label searches start from the present place in program memory, hence a "local label" such as label 6 in Program 2-4 is used many times within the program. The TI-59 cannot use numeric labels, but uses other function keys as labels, e.g., "sin," "fix," etc. There are 62 such keys available for labels. The TI-59 always starts label searches from the top of the program, hence, a given label can only be used once within the program.

The TI-59 is internally set up to be most efficient, time wise, when jumps and branches are made to line numbers rather than to labels. The HP-67/97 appears to be as fast in a label search as the TI-59 is in a line number search. The HP-67/97 cannot go to a specified line number under program control, hence, it is restricted to label searches only. There is a simple program trick shown on page IV-98 of the TI-59 owner's manual where a program is initially written with labels, and the label calls have "NOP" statements following so the program can easily be modified for line number addressing after the program is debugged and complete.

Care should be exercised when translating program coding containing rectangular-to-polar (→P) and polar-to-rectangular (→R) conversions as the TI-59 and HP-67/97 operate on the variables in opposite manner. The HP-67/97 takes the x and y coordinates from the x and y registers and places the magnitude and angle equivalents back into the x and y registers respectively for the →P conversion, and vice-versa for the →R conversion. The TI-59 uses the t and x registers for the two variables, and takes the x and y coordinates from the t and x registers and places the equivalent magnitude and angle back into the t and x registers respectively for the →P conversion, and vice-versa for the →R conversion. Both machines display the contents of the x register, so the TI-59 will display angle or y coordinate whereas the HP-67/97 will display magnitude or x coordinate after respective →P or →R conversions.

To guide the reader in this translation, several programs in this book have been translated into the TI-59 language. These programs have user instructions, examples, and program coding in both languages. Program 1-1 has been flowcharted in addition to provide a common point of reference between the two program listings.

The preceding paragraphs mention anomalies in the TI-59 language. The HP-67/97 language has its idiosyncracies also. Reading the program listings, one will notice some "non-standard" program coding. The prime consideration was to fit the algorithm into the program memory. Within this constraint, the program coding was selected to minimize program execution time whenever possible. Numeric entries within the body of the program are to be avoided, and should be recalled from register storage. Entry of each numeric digit requires 72 milliseconds to execute while a register recall only requires 35 milliseconds typically. Numeric entries such as "10," "100," or any other power of 10 should be entered as a power of ten through the "EEX" key. The number "1" should be entered as "EEX" alone and requires only 48 milliseconds to execute. Similarly, the "CLX" function will result in a zero in the display, and only requires 30 milliseconds to execute. Multiplication of a number by two (2, x) requires 179 milliseconds to execute, while addition of a number to itself (ENT ↑, +) requires 82 milliseconds execution time and yields the same result. Register arithmetic is executed faster than stack arithmetic when the register recalls are considered, and register arithmetic can save program steps. Whenever the algorithm allows, subroutine calls should be minimized as they typically require 240 milliseconds for the label search and return. Likewise unconditional jumps such as GTOA require 160 milliseconds for the label search typically. By paying attention to small details such as these, the program

execution time can be shortened considerably expecially when iteration or looping is required. For more information on execution times and programming hints with the HP-67/97, see "Better Programming on the HP-67/97" edited by William Kolb, John Kennedy, and Richard Nelson, and available from the PPC Club (new name for the HP-65 Users Club), 2541 W. Camden Place, Santa Ana, Calif. 92704.

Even though the program coding has been chosen for minimum execution time, the program LNAP may require more than a minute of computation time before output is provided when the number of branches is large. Likewise, the same time requirement may exist for the filter programs when the filter order is large.

An attempt has been made to choose self-explanatory label descriptions for the user definable keys; hence, once familiar with a particular program, the user need only refer to the magnetic card label markings to run the program.

To restate a point made in the preface of this book, it is not possible to include programs and descriptions covering all areas of engineering analysis and design. The programs herein are only representative of areas in networks and circuits (the terms "networks" and "circuits" may be used interchangeably). The 39 programs contained in this book have been selected from the author's library, and have proved to be quite useful to the author; hopefully, they will prove equally useful to the reader.

The program description not only shows the equations used by the program, but gives a reference, or has a derivation of the equations so these programs may serve as a base for the generation of other related programs as may be needed by the reader for his or her particular application.

Because the programs herein cover several different disciplines in electrical engineering, a problem with nomenclature arises. To the control systems oriented engineer, the term "transfer function" implies system output divided by system input. On the other hand, to the filter design engineer, "transfer function" implies system input divided by system output, or the reciprocal of the control system engineer's definition. To avoid confusion, the term "transmission function" is used to mean system output divided by input and "transfer function" is used to mean system input divided by output. This convention will be followed throughout the book.

The appendix has a list of a list of abbreviations used, along with the bibliography give the reader an easily found place to go should confusion or uncertainty to variable or abbreviation meaning arise.

HANDBOOK OF ELECTRONIC DESIGN AND ANALYSIS PROCEDURES USING PROGRAMMABLE CALCULATORS

Part 1

NETWORK ANALYSIS

PROGRAM 1-1 LOSSY TRANSMISSION LINE INPUT IMPEDANCE.

Program Description and Equations Used

This program uses Eq. (1-1.1) to determine the complex input impedance, Z_s, of a lossy transmission line of length ℓ, loaded with a complex impedance Z_r, and having a characteristic impedance Z_o, an attenuation constant α_{dB} in dB per unit length, and a phase constant β in radians per unit length (or velocity of propagation C_m). For solid dielectric cables, C_m is typically 1/2 to 2/3 the free-space speed of light, and is approximated by Eq. (1-1.9) for low loss coaxial cables, or calculated from Eqs. (1-1.5) and (1-1.6) if the cable impedance and admittance per unit length are known at the operating frequency. The unit of length has purposely not been given because it is to be selected by the user. As long as the same length unit is used throughout, length will cancel out of Eq. (1-1.1). Figure 1-1.1 shows the general circuit topology.

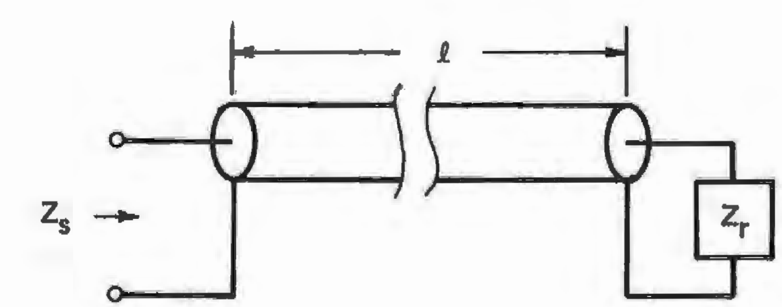


Figure 1-1.1 Transmission line setup.

The equation that describes the problem is:

$$Z_s = Z_o \frac{1 + \rho e^{-2\gamma \ell}}{1 - \rho e^{-2\gamma \ell}} \quad (1\text{-}1.1)$$

where ρ is the reflection coefficient and γ is the propagation function. These quantities are given by the following equations:

$$\rho = \frac{Z_r/Z_o - 1}{Z_r/Z_o + 1} \quad (1\text{-}1.2)$$

$$\gamma = \alpha + j\beta \quad (1\text{-}1.3)$$

$$\alpha = (\alpha_{db})/(20 \log e) \quad (1\text{-}1.4)$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{C_m} \quad (1\text{-}1.5)$$

If the per unit length series impedance, $\bar{R} + j\omega\bar{L}$, and shunt admittance, $\bar{G} + j\omega\bar{C}$, are available at the frequency of operation, then the propagation function is given by:

$$\gamma = \sqrt{(\bar{R} + j\omega\bar{L})(\bar{G} + j\omega\bar{C})} \quad (1\text{-}1.6)$$

If Z_r is desired in terms of Z_s, Z_r is replaced by Z_s in Eq. (1-1.1), Z_s is replaced by Z_r in Eq. (1-1.2), and ℓ is replaced by $-\ell$ in Eq. (1-1.1).

This quasi-symmetrical property allows the use of the same program to calculate the transmission line input impedance with a complex load by using a positive line length, or to calculate the complex load that will provide a specified input impedance by using a negative line length.

A duality also exists with Eq. (1-1.1) and Eq. (1-1.2). The same equation form holds for the transmission line input or output admittance providing each Z is replaced by the corresponding Y, i.e., $Y_s = 1/Z_s$, $Y_r = 1/Z_r$, and $Y_o = 1/Z_o$. The admittance forms of Eqs. (1-1.1) and (1-1.2) are as follows:

$$Y_s = Y_o \frac{1 + \rho' e^{-2\gamma \ell}}{1 - \rho' e^{-2\gamma \ell}} \quad (1\text{-}1.7)$$

where

$$\rho' = \frac{Y_r/Y_o - 1}{Y_r/Y_o + 1} \quad (1\text{-}1.8)$$

$$(\rho' = -\rho)$$

Because the equation form is the same, the program will work with admittances as well as impedances.

In this HP-67/97 program, keys "A" through "E" and "a" through "c" on the calculator have a dual function role. Execution of these keys following a data entry from the keyboard is interpreted as data input by the program, and the numeric entry is stored. Execution of these keys following a nonnumeric entry, or following the "e" (clear) key is interpreted as an output request, and the currently stored values are printed (HP-97 only) and displayed. This feature cannot be translated into the TI-59 program.

The data required by the program is entered in either cartesian (real and imaginary) or polar (magnitude and angle) form through keys "b" and "c," or "B" and "C" respectively. On large coax cables such as underwater telephone cable, both the cable attenuation and phase constants are provided as a function of frequency by the manufacturer, and are loaded into the program using the units of dB per unit length and radians per unit length respectively. If β is unknown, it can be calculated from the velocity of propagation in the transmission line. If the transmission line has less than 1 dB loss in the length being used, and is of coaxial construction, the velocity in the medium (phase velocity) may be approximated by

$$C_m \approx \frac{\text{speed of light in free space}}{\sqrt{\varepsilon_r \mu_r}} \quad (1\text{-}1.9)$$

where ε_r and μ_r are the relative dielectric constant and relative permeability of the cable dielectric and conductors respectively. For cables constructed of nonmagnetic parts, or for cables with a steel strength member within the center conductor of the cable and operating at frequencies where the skin effect keeps currents from flowing within the strength member, the relative permeability, μ_r becomes unity.

User Instructions

HP-67/97 PROGRAM

LOSSY TRANSMISSION LINE INPUT IMPEDANCE

$f \uparrow C_m$	Re $Z_o \uparrow$ Im Z_o	Re $Z_r \uparrow$ Im Z_r	° compute $\|Z_s\|, \measuredangle Z_s$	clear entry mode
$\alpha_{dB} \uparrow \beta \uparrow f$	$\|Z_o\| \uparrow \measuredangle Z_o$	$\|Z_r\| \uparrow \measuredangle Z_r$	$\|\rho\|, \measuredangle\rho$	ℓ

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Enter transmission line parameters			
	line loss in dB/unit length	α_{dB}	↑	
	phase constant in radians/unit length	β	↑	
	frequency in hertz	f	A	C_m
	If velocity of propogation is known instead of phase constant, enter dummy value of 1 for phase constant in step 3 above, then			
	Enter frequency in hertz	f	↑	
	Enter propagation velocity*	C_m	f A	β
	* note: The units of length must be consistent throughout the data, i.e., all in meters, or feet, or miles, etc.			
3	Enter the transmission line characteristics at the chosen analysis frequency			
	magnitude of Z_o in ohms	$\|Z_o\|$	↑	
	phase angle of Z_o in degrees	$\measuredangle Z_o$	B	
	OR			
	real part of Z_o in ohms	Re Z_o	↑	
	imaginary part of Z_o in ohms	Im Z_o	f B	
4	Enter load impedance			
	magnitude of load impedance in ohms	$\|Z_r\|$	↑	
	phase angle of load impedance in degrees	$\measuredangle Z_r$	C	
	OR			
	real part of load impedance in ohms	Re Z_r	↑	
	imaginary part of load impedance in Ω	Im Z_r	f C	

User Instructions

LOSSY TRANSMISSION LINE INPUT IMPEDANCE

HP-67/97

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
5	Enter transmission line length	$\pm \ell$	E	
	$+\ell$ to calculate Z_s given Z_r			
	$-\ell$ to calculate Z_r given Z_s			
6	Optional, printout or enter refl coef			
	ρ entry	$\|\rho\| \uparrow \measuredangle\rho°$	D	
	ρ printout		D	$\|\rho\|, \measuredangle\rho°$
	Of the three variables Z_o, Z_r, & ρ either Z_o & Z_r or Z_o & ρ are required.			
7	Compute $\|Z_s\|, \measuredangle Z_s$ (length positive)		f D	$\|Z\|, \measuredangle Z°$
	$\|Z_r\|, \measuredangle Z_r$ (length negative)			
8	To clear input mode and initialize program		f E	
9	To review input data		f E	
			f A	f , C_m
			A	α_{dB}, β, f
			f B	ReZ_o, ImZ_o
			B	$\|Z_o\|, \measuredangle Z_o°$
			f C	ReZ_r, ImZ_r
			C	$\|Z_r\|, \measuredangle Z_r°$
			D	$\|\rho\|, \measuredangle\rho°$
			E	ℓ
	NOTE: The angular mode of the program is degrees. All angular data input and output is in degrees with the exception of β. The angular mode should not be changed as program malfunction will occur because of R→D and D→R conversions that are used.			

1-1 **User Instructions**

→ TI-59 TRANSLATION ←

LOSSY TRANSMISSION LINE INPUT IMPEDANCE (TI-59)				
C_m	ReZ_o, ImZ_o	ReZ_r, ImZ_r	calculate $\|Z_r\|$, $\measuredangle Z_r$	
α_{dB}, β, f	$\|Z_o\|$, $\measuredangle Z_o$	$\|Z_r\|$, $\measuredangle Z_r$	output $\|\rho\|$, $\measuredangle\rho$	ℓ

TI-59 TRANSLATION

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Load line loss in dB/unit length	α_{dB}	[A]	
	Load line phase constant in rad / unit length	β	[R/S]	
	If C_m, the velocity in the medium, is known instead, load dummy β of 1			
	Load analysis frequency in hertz	f	[R/S]	
3	If C_m is known instead of β, load C_m	C_m	[2nd] [A]	
4	Enter Z_o, the transmission line characteristic impedance in polar or rectangular co-ords			
	polar co-ordinates: magnitude	$\|Z_o\|$, Ω	[B]	
	phase angle	$\measuredangle Z_o$, °	[R/S]	
	rectangular co-ordinates: real part	Re Z_o, Ω	[2nd] [B]	
	imaginary part	Im Z_o, Ω	[R/S]	
5	Enter load impedance at the analysis freq as either polar or rectangular data			
	polar co-ordinates:	$\|Z_r\|$, Ω	[C]	
		$\measuredangle Z_r$, °	[R/S]	
	rectangular co-ordinates:	Re Z_r, Ω	[2nd] [C]	
		Im Z_r, Ω	[R/S]	
6	Load transmission line length	$\pm\ell$	[E]	
	$+\ell$ to calculate Z_s given Z_r			
	$-\ell$ to calculate Z_r given Z_s			
7	Optional: output reflection coefficient		[D]	$\|\rho\|$
			[R/S*]	$\measuredangle\rho°$
8	To calculate Z_s (or Z_r given negative length)		[2nd] [D]	$\|Z\|$, Ω
			[R/S*]	$\measuredangle Z°$
	* If the TI-59 is attached to the PC-100A printer, the second value will be printed without the R/S command.			

Example 1-1.1

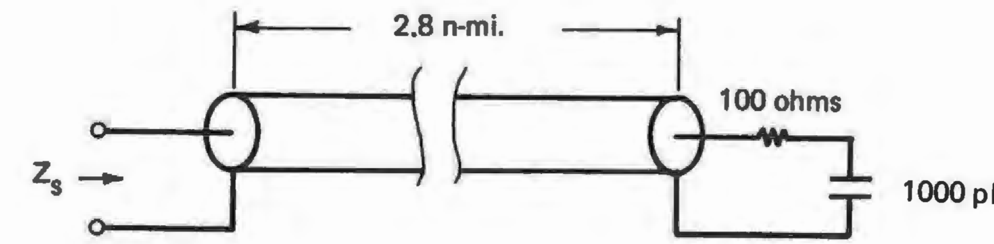


Figure 1-1.2 SD coaxial cable circuit for Ex. 1-1.1.

Type SD underwater telephone coax is to be used at 0.72 MHz. The cable section is 2.8 nautical miles (n-mi.) long and is loaded by a series RC network of 100 ohms and 1000 pF as shown in Fig. 1-1.2. Find the cable input impedance, Z_s, at this frequency.

At 0.72 MHz the electrical parameters of SD coax are:

α_{dB} = 2.070 dB/n-mi.

β = 42.511 radians/n-mi.

Z_o = 44.625 ohms at -0.315 degree

The RC load impedance is:

Re Z_r = 100 ohms

Im Z_r = $-j/(2\pi fC)$ = -j221 ohms

The input impedance of the loaded coax is 66.902 + j11.167 ohms as obtained from using the program and shown in the printout below:

PROGRAM INPUT			PROGRAM OUTPUT		
2.070	ENT↑	α_{dB}, dB/n-mi.		GSBd	calculate Z_s
42.511	ENT↑	β, rad/n-mi.	67.827	***	$\|Z_s\|$, ohms
.72+06	GSBA	frequency, Hz	9.476	***	$\measuredangle Z_s$, degrees
44.265	ENT↑	$\|Z_o\|$, ohms		X⇄Y →R	convert to rect
-.315	GSBB	$\measuredangle Z_o$, degrees	66.902	***	Re Z_s, ohms
100.000	ENT↑	Re Z_r, ohms	11.167	***	Im Z_s, "
-221.000	GSBc	Im Z_r, "		GSBa	calculate C_m
2.800	GSBE	length, n-mi.	720000.000	***	frequency, Hz
			106417.008	***	C_m, n-mi./sec

Example 1-1.2

Using the type SD underwater telephone coax of Example 1-1.1, find the load impedance at 0.72 MHz that will result in an input impedance of 60 + j0 ohms. The length of the coax is 2.8 n-mi. as in the previous example.

When using a lossy cable, a negative real part in Z_r will be required to obtain values of Z_s greatly different than Z_o. Furthermore, if $\alpha\ell$ is greater than 30 dB, the input impedance will be nearly Z_o, independent of the load impedance.

In this example, a negative line length is loaded to use the quasi-symmetric properties of Eqs. (1-1.1) and (1-1.2) for calculating Z_r given Z_s.

The HP-97 printout reproduced next shows a load impedance of 67.396 - j73.338 ohms is required. The equivalent load network is also shown.

PROGRAM INPUT		PROGRAM OUTPUT		
			GSBa	calculate load Z_r
2.070 ENT↑	α_{dB}, dB/n-mi.	99.603	***	$\lvert Z_r \rvert$, ohms
42.511 ENT↑	β, rad/n-mi.	-47.418	***	$\measuredangle Z_r$, degrees
.72+06 GSBA	frequency, Hz			
44.265 ENT↑	$\lvert Z_o \rvert$, ohms		X⇄Y	convert to rect
-.315 GSBB	$\measuredangle Z_o$, degrees		→R	
60.000 ENT↑	Re Z_s, ohms	67.396	***	Re Z_r, ohms
0.000 GSBc	Im Z_s, "	-73.338	***	Im Z_r, "
-2.800 GSBE		2.000	Pi	
			×	calculate
		.72+06	×	equivalent
			×	capacitor
			1/X	
		3.014-09	***	C, farad

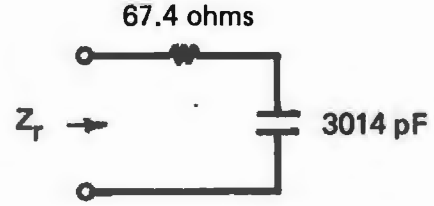


Figure 1-1.3 Equivalent load network.

Example 1-1.3, TI-59 Program Example

This example is the same as Example 1-1.2 where the problem is to determine load impedance, Z_r, that results in an input impedance, Z_s, of 60 + j0 ohms. The line length is 2.8 n-mi. Because Z_r is to be calculated given Z_s, a negative line length is used. The PC-100A printer output is shown below.

PROGRAM INPUT

2.07	α_{dB}, dB/n-mi.
42.511	β, rad/n-mi.
7.2 05	frequency, Hz
106417.0079	C_m (output), n-mi./sec
44.265	$\lvert Z_o \rvert$, ohms
-0.315	$\measuredangle Z_o$, degrees
60.	Re Z_s, ohms
0.	Im Z_s, "
-2.8	line length, n-mi.

PROGRAM OUTPUT

.1509385583	$\lvert\rho\rvert$, dimensionless
1.019762234	$\measuredangle\rho$, degrees
99.60303649	$\lvert Z_r \rvert$, ohms
-47.41754913	$\measuredangle Z_r$, degrees

Note: the PC-100A printer will not print the mnemonic representing the input key. The HP-97 does this automatically when in the "norm" mode.

PROGRAM FLOW DIAGRAM

LABEL A, Input/Output of α, β, and frequency.

Input data format: α, β, and freq

↓

Temporarily store the frequency in register A, set flag 0 to indicate label A as subroutine 1 entry, clear flag 2 to indicate R→P not needed, place 0 in X register as I/O index.

↓

Go sub subroutine 1 to store I/O index and I/O the data in polar format.

↓

Calculate $C_m = (2\pi f)/\beta$ and store in secondary register 5.

↓

Go to label 7 for paper space and return.

LABEL B, Input/Output of Z_o, the characteristic impedance, in polar format.

Input data format: $|Z_o|$, $\measuredangle Z_o$

↓

Place 1 in X register as I/O index and go sub subroutine 1 to store I/O index and store/recall I/O in polar format.

↓

Go to label 6 for data entry test and ρ calculation if new data has been entered.

LABEL C, Input/Output of load impedance, Z_r, in polar format.

Input data format: $|Z_r|$, $\measuredangle Z_r$

↓

Place 2 in X register as I/O index and go sub subroutine 1 to store the I/O index and I/O data in polar format.

↓

Go to label 6 for data entry test and ρ calculation if new data has been entered.

LABEL D, Input/Output of reflection coefficient, ρ, in polar format.

Input data format: $|\rho|$, $\measuredangle \rho$

↓

Place 3 in X register as I/O index and go sub subroutine 1 to store I/O index and I/O data in polar format.

↓

Clear data entry flag 3, space, and return.

PROGRAM FLOW DIAGRAM

LABEL E, Input/Output of the line length

↓

Flag 3 ?

(+) → Go to label 2. → LABEL 2, X→R4, go to label 9. → LABEL 9

(−) → Recall and print R4, go to label 9. → LABEL 9

LABEL 9, Space paper and return control to keyboard.

LABEL a, I/O of frequency and velocity in the medium, C_m.

↓

Place 5 in X register for I/O index and go sub subroutine 1 to store I/O index and store/recall data.

↓

Calculate and store $\beta = (2\pi f)/C_m$.

↓

Space paper and return control to keyboard.

LABEL b, I/O of characteristic impedance, Z_o, in rectangular format.

↓

1 → X for I/O index, gosub 0 to store I/O index, convert data to polar format, and store polar data.

↓

Go to label 6 for data entry test and ρ calculation if new data has been entered.

LABEL c, I/O of load impedance, Z_r, in rectangular format.

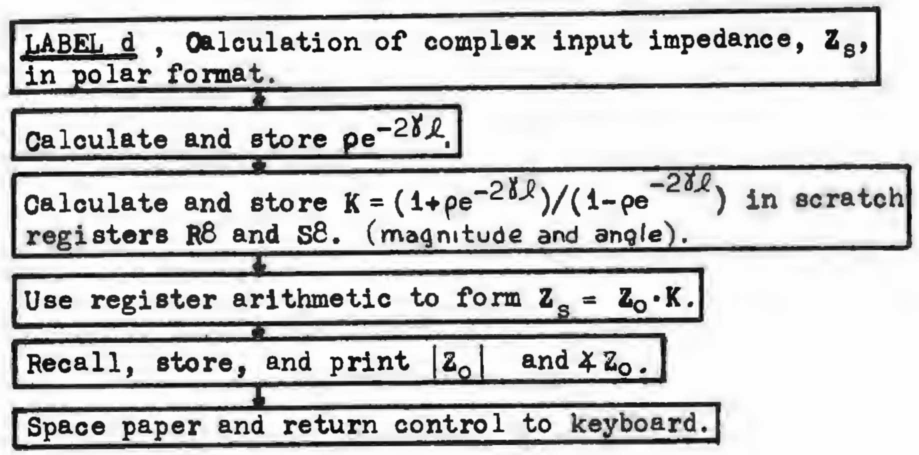
↓

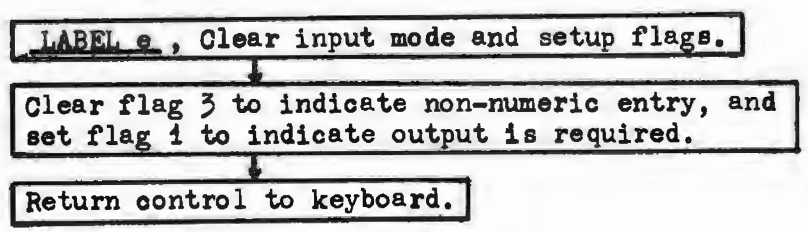
2 → X for I/O index, gosub 0 to store I/O index, convert data to polar format, and store polar data.

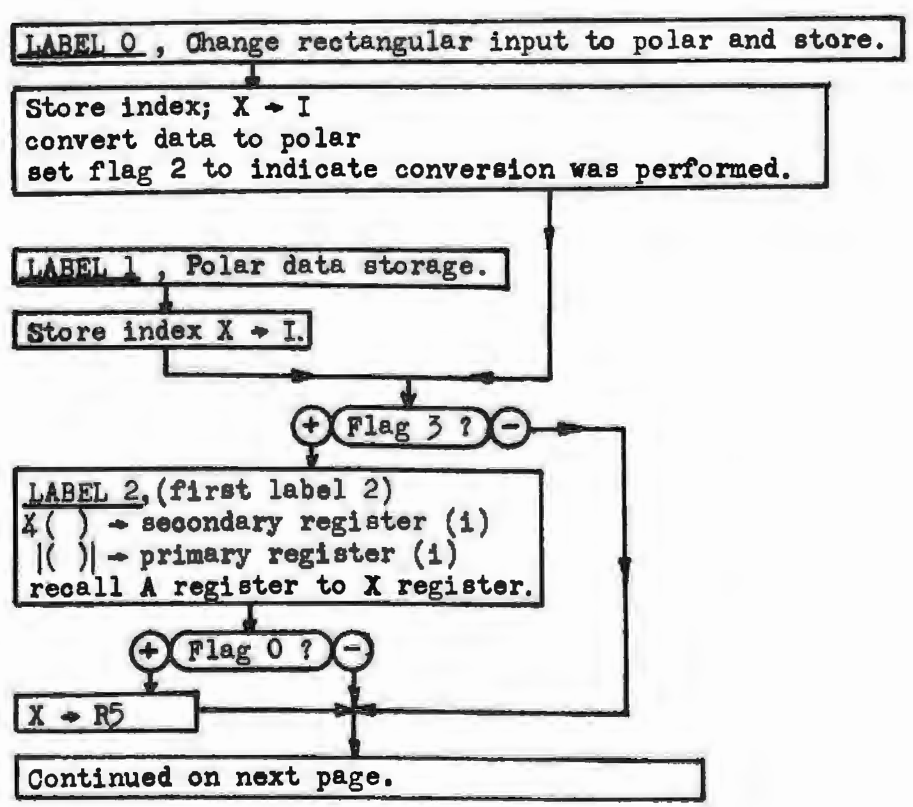
↓

Go to label 6 for data entry test and ρ calculation if new data has been entered.

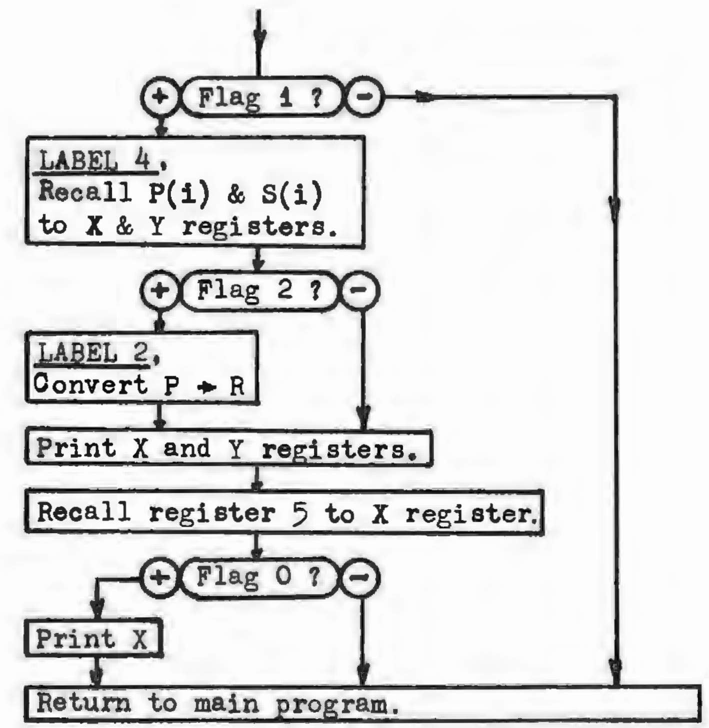
PROGRAM FLOW DIAGRAM

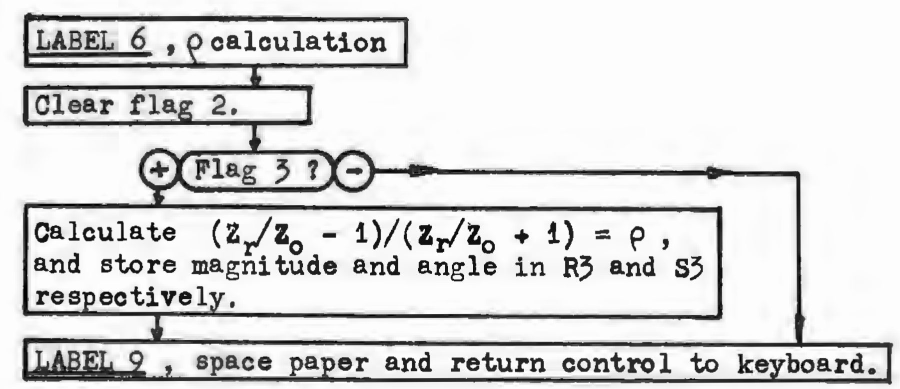






PROGRAM FLOW DIAGRAM





Program Listing I

Step	Key	Comment
001	*LBLA	I/O OF α_{dB}, β, AND FREQ
002	STOA	
003	R↓	
004	SF0	
005	CF2	
006	0	
007	GSB1	
008	CF0	
009	CF3	
010	Pi	
011	Pi	
012	+	
013	RCL5	
014	x	
015	P⇄S	
016	RCL0	
017	÷	
018	STO5	
019	GTO7	
020	*LBLB	I/O OF $\|Z_o\|$, $\measuredangle Z_o$, THE CHARACTERISTIC IMPEDANCE IN POLAR CO-ORDINATES
021	EEX	
022	GSB1	
023	GTO6	
024	*LBLC	I/O OF $\|Z_r\|$, $\measuredangle Z_r$, THE LOAD IMPEDANCE IN POLAR CO-ORDINATES
025	2	
026	GSB1	
027	GTO6	
028	*LBLD	I/O OF $\|\rho\|$, $\measuredangle\rho$, THE COMPLEX REFLECTION COEF IN POLAR CO-ORDINATES
029	3	
030	GSB1	
031	CF3	
032	GTO9	
033	*LBLE	I/O OF THE LINE LENGTH
034	F3?	
035	GTO2	
036	RCL4	
037	GTO8	
038	*LBL2	
039	STO4	
040	GTO9	
041	*LBLa	I/O OF FREQUENCY AND VELOCITY IN THE MEDIUM, C_m
042	5	
043	GSB1	
044	CF3	
045	RCL5	
046	ENT↑	
047	+	
048	Pi	
049	x	
050	P⇄S	
051	RCL5	
052	÷	
053	STO0	
054	GTO7	
055	*LBLb	I/O OF Re Z_o, Im Z_o, THE CHARACTERISTIC IMPEDANCE IN CARTESIAN CO-ORDINATES
056	EEX	
057	GSB0	
058	GTO6	
059	*LBLc	I/O OF Re Z_r, Im Z_r, THE LOAD IMPEDANCE IN CARTESIAN CO-ORDINATES
060	2	
061	GSB0	
062	GTO6	
063	*LBLd	CALCULATION OF $\|Z_s\|$, $\measuredangle Z_s$, THE COMPLEX INPUT IMPEDANCE
064	RCL3	
065	RCL0	
066	RCL4	
067	x	
068	EEX	
069	1	
070	÷	
071	CHS	
072	10^x	$e^{-2\alpha\ell}$
073	x	
074	STOA	$\|\rho\|\cdot e^{-2\alpha\ell} = \|\rho e^{-2\gamma\ell}\|$
075	P⇄S	
076	RCL3	$\measuredangle\rho$
077	RCL0	β, radians/length
078	R→D	β, degrees/length
079	P⇄S	
080	RCL4	ℓ
081	x	
082	ENT↑	
083	+	$2\beta\ell$
084	−	$\measuredangle\rho - 2\beta\ell = \measuredangle(\rho e^{-2\gamma\ell})$
085	RCLA	
086	→R	
087	STOA	$Re(\rho e^{-2\gamma\ell})$
088	EEX	
089	+	$1 + Re(\rho e^{-2\gamma\ell})$
090	X⇄Y	
091	STOB	$Im(\rho e^{-2\gamma\ell})$
092	X⇄Y	
093	→P	
094	STO7	$\|1 + \rho e^{-2\gamma\ell}\|$
095	X⇄Y	
096	STO9	$\measuredangle(1 + \rho e^{-2\gamma\ell})$
097	RCLB	
098	CHS	$Im(1 - \rho e^{-2\gamma\ell})$
099	EEX	
100	RCLA	
101	−	$Re(1 - \rho e^{-2\gamma\ell})$
102	→P	
103	ST÷7	
104	X⇄Y	
105	ST-9	
106	RCL1	$\|Z_o\|$
107	STx7	
108	P⇄S	
109	RCL1	$\measuredangle Z_o$
110	P⇄S	
111	ST+9	
112	RCL9	$\measuredangle Z_s$

REGISTERS

0 α, dB/ℓ	1 $\|Z_o\|$	2 $\|Z_r\|$	3 $\|\rho\|$	4 ℓ	5 freq	6 $\measuredangle\rho$, scratch	7 $\pi\|z\|$	8 $\|z\|$	9 $\Sigma\measuredangle z$
S0 β, rad/ℓ	S1 $\measuredangle Z_o$	S2 $\measuredangle Z_r$	S3 $\measuredangle\rho$	S4	S5 C_m	S6	S7	S8 $\measuredangle Z$	S9

A scratchpad	B scratchpad	C scratchpad	D	E	I index

Program Listing II

Step	Key	Comment
113	RCL7	$\|Z_s\|$
114	→R	eliminates neg magnitude
115	→P	
116	STO8	$\|Z_s\|$
117	PRTX	
118	X⇄Y	
119	P⇄S	
120	STO8	$\measuredangle Z_s$
121	P⇄S	
122	GSB8	
123	GTO9	
124	*LBLe	CLEAR INPUT MODE initialize and set flags
125	CF3	
126	SF1	
127	RTN	
128	*LBL0	change rectangular input to polar format
129	STOI	
130	R↓	
131	X⇄Y	
132	→P	
133	X⇄Y	
134	SF2	
135	GTO2	
136	*LBL1	data I/O in polar mode
137	STOI	
138	R↓	
139	*LBL2	
140	F3?	test for input
141	GSB2	goto input routine
142	F1?	test for output
143	GSB4	goto output routine
144	SF1	
145	RTN	
146	*LBL2	input data storage routine (data stored in polar form)
147	SF3	
148	P⇄S	
149	STOi	
150	P⇄S	
151	R↓	
152	STOi	
153	RCLA	
154	F0?	
155	STO5	
156	CF1	
157	RTN	
158	*LBL4	data output routine
159	P⇄S	
160	RCLi	
161	P⇄S	
162	RCLi	
163	F2?	P → R required ?
164	GTO2	
165	PRTX	
166	R↓	
167	PRTX	
168	RCL5	recall frequency
169	F0?	print required ?
170	PRTX	print frequency
171	RTN	
172	*LBL2	convert polar data to rectangular format and print results
173	→R	
174	PRTX	
175	R↓	
176	*LBL8	print and space subroutine
177	PRTX	
178	GTO9	goto space subroutine
179	*LBL6	ρ calculation
180	CF2	
181	F3?	ρ calculation needed ?
182	F3?	
183	GTO9	goto space and return subr
184	P⇄S	
185	RCL2	$\measuredangle Z_r$
186	RCL1	$\measuredangle Z_o$
187	−	$\measuredangle(Z_r - Z_o)$
188	P⇄S	
189	RCL2	$\|Z_r\|$
190	RCL1	$\|Z_o\|$
191	X=0?	exit if $\|Z_o\|$ is zero
192	GTO9	
193	÷	$\|Z_r/Z_o\|$
194	→R	
195	STOA	$Re(Z_r/Z_o)$
196	EEX	
197	−	$Re(Z_r/Z_o - 1)$
198	X⇄Y	
199	STOB	$Im(Z_r/Z_o - 1)$
200	X⇄Y	
201	→P	
202	STO3	$\|Z_r/Z_o - 1\|$
203	X⇄Y	
204	STO6	$\measuredangle(Z_r/Z_o - 1)$
205	RCLB	$Im(Z_r/Z_o)$
206	RCLA	$Re(Z_r/Z_o)$
207	EEX	
208	+	$Re(Z_r/Z_o + 1)$
209	→P	
210	ST÷3	$\|\rho\|$
211	X⇄Y	
212	ST-6	$\measuredangle\rho$
213	RCL6	
214	P⇄S	
215	STO3	
216	*LBL7	P⇄S & space subroutine
217	P⇄S	
218	*LBL9	space and return subr
219	SPC	
220	RTN	

LABELS					FLAGS
A I/O: α_{dB}, β, f	B I/O: $\|\|$, $\measuredangle Z_o$	C I/O: $\|\|$, $\measuredangle Z_r$	D I/O: $\|\|$, $\measuredangle\rho$	E I/O: length	0 label A ?
a I/O: f, C_m	b I/O: ReImZ_o	c I/O: ReImZ_r	d calc. Z_s	e clear input mode	1 input or output
0 P → R	1 I/O index	2 local lbl	3	4 print in polar	2 cartesian data fmt
5	6 calc ρ	7 p⇄s, spc, rt	8 prt, spc, rt	9 spc, rtn	3 data entry

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0		■	DEG	■	FIX	■
1	■		GRAD		SCI	
2		■	RAD		ENG	
3		■			n 2	

1-1 **TI-59 PROGRAM LISTING**

Loc	Code	Key	Comments
000	76	LBL	LOAD α_{dB}
001	11	A	
002	42	STO	store and print
003	00	00	α dB
004	99	PRT	
005	91	R/S	LOAD β
006	42	STO	store and print
007	10	10	β
008	99	PRT	
009	91	R/S	LOAD FREQUENCY
010	42	STO	store and print
011	05	05	frequency
012	99	PRT	
013	65	×	calculate and
014	02	2	store $2\pi f$
015	65	×	
016	89	π	
017	95	=	
018	42	STO	
019	26	26	
020	55	÷	calculate and
021	43	RCL	store C_m
022	10	10	
023	95	=	$C_m = \frac{2\pi f}{\beta}$
024	42	STO	
025	15	15	
026	02	2	set flag 7 if
027	00	0	calculator
028	69	OP	attached to
029	07	07	printer
030	69	OP	
031	19	19	
032	25	CLR	
033	43	RCL	recall C_m and go
034	15	15	to R/S or print
035	71	SBR	routine
036	68	NOP	
037	98	ADV	
038	92	RTN	
039	76	LBL	LOAD C_m
040	16	A'	
041	42	STO	store and print
042	15	15	C_m
043	99	PRT	
044	35	1/X	calculate and
045	65	×	store β
046	43	RCL	$\beta = \frac{2\pi f}{C_m}$
047	26	26	
048	95	=	
049	42	STO	
050	10	10	
051	71	SBR	go to print or R/S
052	68	NOP	routine
053	98	ADV	
054	92	RTN	
055	76	LBL	LOAD $\|Z_o\|$
056	12	B	
057	42	STO	store and print
058	01	01	$\|Z_o\|$
059	99	PRT	
060	91	R/S	LOAD $\measuredangle Z_o$
061	42	STO	store and print
062	11	11	$\measuredangle Z_o$
063	99	PRT	
064	98	ADV	
065	61	GTO	goto ρ calculation
066	70	RAD	subroutine
067	76	LBL	LOAD Re Z_o
068	17	B'	
069	99	PRT	print and store
070	32	X:T	ReZ_o
071	91	R/S	LOAD Im Z_o
072	22	INV	convert to polar
073	37	P/R	
074	42	STO	store $\measuredangle Z_o$
075	11	11	
076	32	X:T	recall and store
077	42	STO	$\|Z_o\|$
078	01	01	
079	98	ADV	go to ρ calculation
080	61	GTO	subroutine
081	70	RAD	
082	76	LBL	LOAD $\|Z_r\|$
083	13	C	
084	42	STO	store and print
085	02	02	$\|Z_r\|$
086	99	PRT	
087	91	R/S	LOAD $\measuredangle Z_r$
088	42	STO	store and print
089	12	12	$\measuredangle Z_r$
090	99	PRT	
091	98	ADV	
092	61	GTO	goto ρ calculation
093	70	RAD	subroutine
094	76	LBL	LOAD Re Z_r
095	18	C'	
096	99	PRT	store and print
097	32	X:T	Re Z_r
098	91	R/S	LOAD Im Z_r
099	99	PRT	print Im Z_r

NOTE: The register assignments are the same as the HP-97 program. Read S0 as R_{10}, and RA as R_{20}, etc. R_{26} - R_{28} are scratchpads

1-1 **TI-59 PROGRAM LISTING**

Loc	Code	Key	Comments
100	98	ADV	
101	22	INV	convert to polar
102	37	P/R	
103	42	STO	store $\measuredangle Z_r$
104	12	12	
105	32	X:T	store $\|Z_r\|$
106	42	STO	
107	02	02	
108	76	LBL	ρ calculation
109	70	RAD	
110	43	RCL	calculate & store:
111	12	12	
112	75	−	$\measuredangle(Z_r - Z_o)$
113	43	RCL	
114	11	11	
115	95	=	
116	32	X:T	
117	43	RCL	calculate & store:
118	02	02	
119	55	÷	$\|Z_r / Z_o\|$
120	43	RCL	
121	01	01	
122	95	=	
123	32	X:T	
124	37	P/R	convert to rect
125	42	STO	store $Im(Z_r/Z_o)$
126	27	27	
127	32	X:T	
128	42	STO	store $Re(Z_r/Z_o)$
129	28	28	
130	75	−	calculate & store:
131	01	1	
132	95	=	$Z_r/Z_o - 1$
133	32	X:T	
134	22	INV	convert to polar
135	37	P/R	
136	42	STO	store $\measuredangle(Z_r/Z_o - 1)$
137	13	13	
138	43	RCL	recall & store:
139	27	27	$Im(Z_r/Z_o + 1)$
140	32	X:T	
141	42	STO	store $\|Z_r/Z_o - 1\|$
142	03	03	
143	43	RCL	
144	28	28	form $Z_r/Z_o + 1$
145	85	+	
146	01	1	
147	95	=	
148	32	X:T	
149	22	INV	convert to polar
150	37	P/R	
151	22	INV	use register
152	44	SUM	arithmetic to form:
153	13	13	$\measuredangle\rho$
154	32	X:T	use register
155	22	INV	arithmetic to form:
156	49	PRD	$\|\rho\|$
157	03	03	
158	92	RTN	rtn to main pgm
159	76	LBL	ρ OUTPUT ROUTINE
160	14	D	
161	43	RCL	recall $\|\rho\|$
162	03	03	
163	71	SBR	goto print or R/S
164	68	NOP	
165	43	RCL	recall $\measuredangle\rho$
166	13	13	
167	71	SBR	goto print or R/S
168	68	NOP	
169	98	ADV	space and return
170	92	RTN	
171	76	LBL	CALCULATE Z_s
172	19	D'	
173	43	RCL	form $\alpha\ell$ in dB
174	00	00	
175	65	×	
176	43	RCL	
177	04	04	
178	55	÷	convert to nepers
179	53	(	
180	01	1	
181	22	INV	
182	23	LNX	
183	28	LOG	
184	65	×	
185	01	1	
186	00	0	
187	54	)	
188	95	=	
189	94	+/-	calculate:
190	22	INV	$e^{-2\alpha\ell}$
191	23	LNX	
192	65	×	calculate & store:
193	43	RCL	
194	03	03	$\|\rho e^{-2\alpha\ell}\|$
195	95	=	
196	32	X:T	
197	43	RCL	recall $\measuredangle\rho$
198	13	13	
199	75	−	

1-1 TI-59 PROGRAM LISTING

200	53	(	
201	43	RCL	form $\beta\ell$ in radians
202	10	10	
203	65	×	
204	43	RCL	
205	04	04	
206	65	×	
207	03	3	form $2\beta\ell$ in degrees
208	06	6	
209	00	0	
210	55	÷	
211	89	π	
212	54	)	
213	95	=	form $\angle\rho - 2\beta\ell$
214	37	P/R	convert to rect
215	42	STO	store $\mathrm{Im}(\rho e^{-2\gamma\ell})$
216	21	21	
217	32	X⇌T	
218	42	STO	store $\mathrm{Re}(\rho e^{-2\gamma\ell})$
219	20	20	
220	85	+	form:
221	01	1	$1 + \rho e^{-2\gamma\ell}$
222	95	=	
223	32	X⇌T	
224	22	INV	convert to polar
225	37	P/R	
226	42	STO	store $\angle(1 + \rho e^{-2\gamma\ell})$
227	09	09	
228	32	X⇌T	
229	42	STO	store $\left\|1 + \rho e^{-2\gamma\ell}\right\|$
230	07	07	
231	01	1	form and store:
232	75	−	
233	43	RCL	$\mathrm{Re}(1 - \rho e^{-2\gamma\ell})$
234	20	20	
235	95	=	
236	32	X⇌T	
237	43	RCL	form:
238	21	21	$\mathrm{Im}(1 - \rho e^{-2\gamma\ell})$
239	94	+/−	
240	22	INV	convert to polar
241	37	P/R	
242	22	INV	
243	44	SUM	divide to memory
244	09	09	
245	32	X⇌T	
246	22	INV	subtract from
247	49	PRD	memory
248	07	07	
249	43	RCL	recall $\|Z_o\|$

250	01	01	use register arith
251	49	PRD	to form $\|Z_r\|$
252	07	07	
253	43	RCL	use register arith
254	11	11	to form $\angle Z_r$
255	44	SUM	
256	09	09	
257	43	RCL	recall $\|Z_r\|$
258	07	07	
259	32	X⇌T	
260	43	RCL	recall $\angle Z_r$
261	09	09	
262	37	P/R	eliminate negative
263	22	INV	magnitude
264	37	P/R	
265	42	STO	store $\angle Z$
266	18	18	
267	32	X⇌T	store $\|Z\|$
268	42	STO	
269	08	08	
270	71	SBR	goto print or R/S
271	68	NOP	
272	43	RCL	recall $\angle Z$
273	18	18	
274	71	SBR	goto print or R/S
275	68	NOP	
276	98	ADV	space & return
277	92	RTN	
278	76	LBL	print or R/S
279	68	NOP	subroutine
280	87	IFF	jump if flag 7 set
281	07	07	
282	38	SIN	
283	91	R/S	stop & await start
284	92	RTN	return to main pgm
285	76	LBL	
286	38	SIN	
287	99	PRT	print
288	92	RTN	rtn to main program
289	76	LBL	LOAD LINE LENGTH
290	15	E	
291	42	STO	store line length
292	04	04	
293	99	PRT	print line length
294	98	ADV	
295	92	RTN	rtn to keyboard

PROGRAM 1-2 VOLTAGE ALONG A LOSSY LOADED TRANSMISSION LINE.

Program Description and Equations Used

This program calculates the voltage V(x) in dBV, at any distance, x, along a doubly loaded transmission line (a line with terminating Y's or Z's at both ends). Both the source and load impedances are allowed to be complex quantities. This program is parasitic to Program 1-1, and that program must be run first to properly load the registers for this program. The same line length and units must be used with both programs.

Given a section of transmission line of length ℓ (Fig. 1-2.1) which may be a coax as shown, or open wire line, stripline, microstrip, or other, the input impedance, Z_s, can be expressed in terms of the load impedance, Z_r, and the cable parameters as given by Eqs. (1-1.1) and (1-1.2). With the input impedance, Z_s known, and given the transmitter source impedance, Z_t, the voltage at the input of the transmission line, V_s, is given by:

$$V_s = V_t \left[\frac{Z_s}{Z_s + Z_t}\right] \qquad (1\text{-}2.1)$$

where Z_s is given by Eq. (1-1.1).

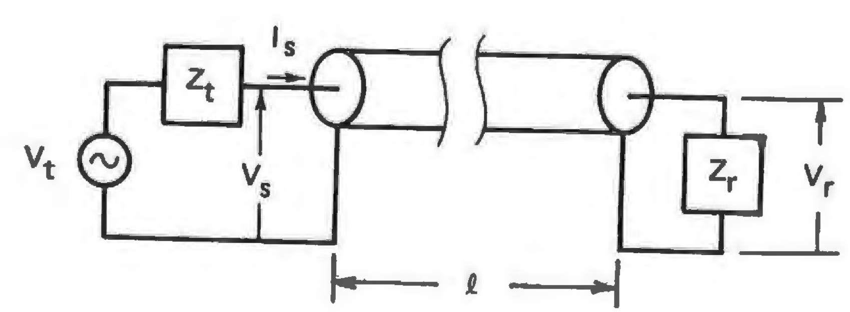


Figure 1-2.1 Transmission line circuit topology.

The voltage and current distribution of the transmission line can be written in terms of the voltage and current at any point along the transmission line as the reference. Most commonly, the voltage at the receiving end is taken as the reference, but for this problem, the voltage and current at the transmitting end are more convenient references. The voltage at any distance, x, from the transmitting end is given by Eq. (1-2.2), where the reflection coefficient at the transmitter is designated ρ_t and is defined by Eq. (1-2.3). The derivation of Eq. (1-2.2) is given later.

$$V(x) = \frac{V_s}{1+\rho_t} \cdot \left[e^{-\gamma x} + \rho_t e^{\gamma x}\right] \tag{1-2.2}$$

$$\rho_t = \frac{Z_s/Z_o - 1}{Z_s/Z_o + 1} \tag{1-2.3}$$

In Eq. (1-2.2), γ is as defined in Eq. (1-1.3). With these equations in mind, the program operation is now described (Program 1-1 has already calculated and stored Z_s using Eq. (1-1.1)).

The routines under labels "A," "a," and "B" provide for data entry and storage. All impedances are stored in polar form; hence, impedances entered in cartesian form (real and imaginary) under label "a" are converted to polar form and stored using the routine under label "A," which is the polar impedance entry and storage routine. The routine under label "B" causes the source voltage strength in volts to be stored.

Label "E" is the start of the data output routine. On the first execution of label "E" after program loading and data entry, ρ_t is calculated and stored. Flag 2 is tested on each execution of label "E" to determine if the reflection coefficient calculation is needed (ρ_t). Since flag 2 is test cleared, and is only set by card loading, the ρ_t calculation is skipped after the first execution of label "E."

Following the ρ_t calculation decision, is a routine to evaluate Eq. (1-2.2) without the V_s term (lines 050 and 096 in the program listing). V_s is calculated using Eq. (1-2.1) in lines 097 through 118 and combined with the results of Eq. (1-2.2) in lines 119 to 125. The output is provided as magnitude (in dBV) of V(x) and its angle.

Label 9 is a space and return subroutine used by labels "a," "A," "B," and "E."

1-2

User Instructions

VOLTAGE ALONG A LOSSY LOADED TRANSMISSION LINE				
load $ReZ_t \uparrow ImZ_t$ / load $\lvert Z_t\rvert \uparrow \measuredangle Z_t$	load source voltage			load dist from xmit & calc: $\lvert V(x)\rvert_{dB}$, $\measuredangle V(x)$

2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
	This program is to be used in conjunction with Program 1-1. Run that program first using the frequency, cable parameters, and total line length which are germane to this program			
1	Load and run Program 1-1			
2	Load both sides of Program 1-2 magnetic card			
3	Load transmitter output impedance			
	a) If data is in cartesian coordinates:			
	real part of impedance in ohms	$Re\ Z_t$	[ENT]	
	imaginary part of impedance in ohms	$Im\ Z_t$	[A]	
	or			
	b) If data is in polar coordinates:			
	magnitude of impedance in ohms	$\lvert Z_t\rvert$	[ENT]	
	angle of impedance in degrees	$\measuredangle Z_t$	[f] [A]	
4	Load source voltage of transmitter in volts	V_t	[B]	
5	Load length between transmitter and analysis point using the same units as used with Program 1-1	x	[E]	$20 \log\lvert V(x)\rvert$ $\measuredangle V(x)°$
6	Go back to step 5 for another case			

Example 1-2.1

Given the coax cable with source and load impedances as shown in Fig. 1-2.2, find the voltages on the cable at the transmitting end, the receiving end, and 1 n-mi. from the transmitting end.

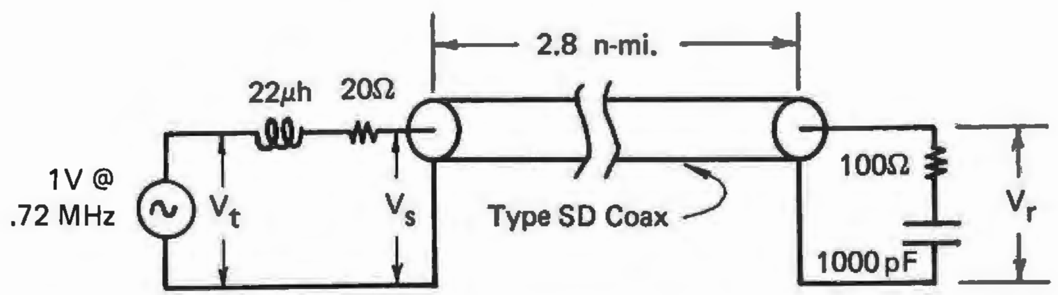


Figure 1-2.2 Doubly loaded coaxial cable for Ex. 1-2.1.

At 0.72 MHz, the characteristics of the SD coax cable are:

α_{dB} = 2.070 dB/n-mi.
β = 42.511 radians/n-mi.
Z_o = 44.265 Ω @ -0.315 degree

At the same frequency, the complex source and load impedances are:

Re Z_t = 20 ohms
Im Z_t = $j2\pi fL$ = j99.53 ohms

Re Z_r = 100 ohms
Im Z_r = $-j/(2\pi fC)$ = -j221 ohms

Since this program is parasitic to Program 1-1, that program is run first with the line length required here (2.8 n-mi.). The printout from that program is included here for clarity.

HP-97 printout for Example 1-2.1

First, Program 1-1 is run to calculate and store Z_s and to load the registers.

2.070	ENT↑	load α_{dB} in dB/n-mi.
42.511	ENT↑	load β in radians/n-mi.
.72+06	GSBA	load frequency in hertz
44.265	ENT↑	load $\|Z_o\|$ in ohms
-.315	GSBB	load $\measuredangle Z_o$ in degrees
100.000	ENT↑	load Re Z_r in ohms
-221.000	GSBc	load Im Z_r in ohms
2.800	GSBE	load line length in nautical miles
	GSBd	calculate Z_s (will be automatically stored)
67.827	***	$\|Z_s\|$, ohms
9.476	***	$\measuredangle Z_s$, degrees

Second, load and run this program.

20.00	ENT↑	load Re Z_t
99.53	GSBc	load Im Z_t
1.00	GSBE	load source voltage in volts
0.00	GSBE	load line length to transmitting end and start
-6.34	***	20 log $\|V_s\|$, dBV
-42.39	***	$\measuredangle V_s$, degrees
2.80	GSBE	load line length to receiving end and start
-8.54	***	20 log $\|V_r\|$, dBV
-34.96	***	$\measuredangle V_r$, degrees
1.00	GSBE	load line length to 1 n-mi. from xmit end and start
-12.96	***	20 log $\|V(x)\|$, dBV
22.16	***	$\measuredangle V(x)$, degrees

Derivation of Equations Used

A transmission line provides a conduit for the propagation of electrical power. If the transmission line is not terminated in the characteristic impedance of the line, Z_o, then not all of the power that propagates down the line is absorbed in the termination, and thus some is reflected into the line and propagates back to the source. The "reflection coefficient," ρ, is a measure of the amount of power that is reflected. A reflection coefficient of zero ($\rho = 0$) implies no power is reflected, and all of it is absorbed by the load. When $\rho = \pm 1$, all the power is reflected. The reflection coefficient in terms of the characteristic impedance (Z_o) and the load impedance (Z_r) is given by Eq. (1-1.2).

If the transmission line is doubly terminated, then there will be a reflection coefficient for both ends, and Eq. (1-1.2) is used with Z_r replaced by Z_s, the cable input impedance at the transmitter end. This is the transmitter reflection coefficient and is designated ρ_t. The receiver reflection coefficient is left unsubscripted.

The power propagates along the transmission line as a voltage wave and a current wave. Considering both the voltage wave from the transmitter directly, and the reflected wave from the receiver, there exist points along the cable where these waves are in phase, and constructively add together; while there are other points where the waves are 180° out of phase and produce a voltage null.

Reference [43] (chapters 8 and 9) contains the solution to the wave equation for voltage and current waves traveling along a transmission line. The voltage and current along the transmission line can conveniently be expressed in terms of hyperbolic functions and a reference voltage and current taken at any point on the line. If x represents the distance from the transmitter (or source) to the point under observation, then the voltage and current (V(x) and I(x)) at this point are:

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} \{\text{Cosh } (\gamma x)\} & \{-Z_o \text{ Sinh } (\gamma x)\} \\ \{\frac{-1}{Z_o} \text{ Sinh } (\gamma x)\} & \{\text{Cosh } (\gamma x)\} \end{bmatrix} \cdot \begin{bmatrix} V_s \\ I_s \end{bmatrix} \quad (1\text{-}2.4)$$

where the hyperbolic functions are defined by:

$$\text{Sinh } (\gamma x) = \frac{e^{\gamma x} - e^{-\gamma x}}{2} \quad (1\text{-}2.5)$$

$$\text{Cosh } (\gamma x) = \frac{e^{\gamma x} + e^{-\gamma x}}{2} \quad (1\text{-}2.6)$$

Remembering that $I_s = V_s/Z_s$, and using the transmitter reflection coefficient defined by:

$$\rho_t = \frac{Z_s/Z_o - 1}{Z_s/Z_o + 1} \quad (1\text{-}2.3)$$

Equation (1-2.4) may be solved for V(x) yielding:

$$V(x) = \frac{V_s}{1 + \rho_t} \cdot \left[e^{-\gamma x} + \rho_t \, e^{\gamma x} \right] \quad (1\text{-}2.2)$$

Program Listing I

Step	Key	Comment
001	*LBLa	LOAD transmitter output Z
002	X⇄Y	in cartesian coordinates
003	→P	
004	X⇄Y	
005	*LBLA	LOAD transmitter output Z
006	P⇄S	in polar coordinates
007	STO7	
008	X⇄Y	
009	STO6	
010	P⇄S	
011	GTO9	goto space and return subr
012	*LBLB	LOAD source voltage in volts
013	P⇄S	
014	STO9	
015	P⇄S	
016	GTO9	goto space and return subr
017	*LBLE	LOAD distance, x, from
018	STO4	xmit and calculate V(x)
019	F2?	calculate ρ_t on the first
020	F2?	execution of label E
021	GTO1	goto V(x) calculation
022	P⇄S	ρ_t calculation routine
023	RCL8	$\angle Z_s$
024	RCL1	$\angle Z_o$
025	P⇄S	
026	-	
027	RCL8	$\|Z_s\|$
028	RCL1	$\|Z_o\|$
029	÷	
030	→R	
031	STOA	$Re(Z_s/Z_o)$
032	EEX	
033	-	$Re(Z_s/Z_o - 1)$
034	X⇄Y	
035	STOB	$Im(Z_s/Z_o) = Im(Z_s/Z_o - 1)$
036	X⇄Y	
037	→P	
038	STO3	$\|Z_s/Z_o - 1\|$
039	X⇄Y	
040	STO6	$\angle(Z_s/Z_o - 1)$
041	RCLB	$Im(Z_s/Z_o) = Im(Z_s/Z_o + 1)$
042	RCLA	$Re(Z_s/Z_o)$
043	EEX	
044	+	$Re(Z_s/Z_o + 1)$
045	→P	
046	ST÷3	$\|\rho_t\|$
047	X⇄Y	
048	ST-6	$\angle \rho_t$
049	*LBL1	V(x) calculation routine
050	RCL4	ℓ
051	RCL0	α_{dB}
052	×	
053	2	
054	0	
055	÷	$\alpha\ell / \ln 10$, nepers
056	10^x	$e^{\alpha\ell}$
057	STOA	
058	RCL3	
059	×	$\|\rho_t e^{\gamma\ell}\|$
060	RCL6	$\angle \rho_t$
061	P⇄S	
062	RCL0	β, radians/length
063	P⇄S	
064	R→D	β, degrees/ length
065	RCL4	
066	×	
067	STOB	$\beta\ell$, degrees
068	+	$\beta\ell + \angle\rho_t$, degrees
069	X⇄Y	
070	→R	
071	STO7	$Re(\rho_t e^{\gamma\ell})$
072	X⇄Y	
073	STO9	$Im(\rho_t e^{\gamma\ell})$
074	RCLB	calculate $e^{-\gamma\ell}$ in real and
075	CHS	imaginary parts
076	RCLA	
077	1/X	
078	→R	
079	ST+7	continue numerator calc of
080	X⇄Y	(1-2.4) using reg arith
081	ST+9	
082	RCL9	convert numerator to polar
083	RCL7	coordinates
084	→P	
085	STO7	$\|e^{-\gamma\ell} + \rho_t e^{\gamma\ell}\|$
086	X⇄Y	
087	STO9	$\angle(e^{-\gamma\ell} + \rho_t e^{\gamma\ell})$
088	RCL6	calculate 1 + ρ_t in polar
089	RCL3	coordinates
090	→R	
091	EEX	
092	+	
093	→P	
094	ST÷7	divide 1 + ρ_t into
095	X⇄Y	numerator
096	ST-9	
097	P⇄S	calculate V_s from V_t
098	RCL8	$\angle Z_s$
099	P⇄S	
100	ST+9	
101	RCL8	$\|Z_s\|$
102	ST×7	
103	→R	
104	STOA	Re Z_s
105	X⇄Y	
106	STOB	Im Z_s
107	P⇄S	
108	RCL7	form $Z_s + Z_t$
109	RCL6	
110	P⇄S	

REGISTERS

0 α_{dB}	1 $\|Z_o\|$	2 $\|Z_r\|$	3 $\|\rho_t\|$	4 ℓ	5 freq	6 $\angle\rho_t$	7 scratch	8 $\|Z_s\|$	9 scratch
S0 β	S1 $\angle Z_o$	S2 $\angle Z_r$	S3 $\angle\rho$	S4	S5 Ω_m	S6 $\|Z_t\|$	S7 $\angle Z_t$	S8 $\angle Z_s$	S9 V_t
A scratchpad	B scratchpad	C scratchpad	D 20log e	E 2π	I index				

Program Listing II

NOTE FLAG SET STATUS

Step	Key	Comment
111	→R	
112	RCLA	
113	+	$Re(Z_s + Z_t)$
114	X⇄Y	
115	RCLB	
116	+	$Im(Z_s + Z_t)$
117	X⇄Y	
118	→P	
119	ST÷7	$\|Z_s + Z_t\|$
120	X⇄Y	
121	ST-9	$\angle(Z_s + Z_t)$
122	P⇄S	
123	RCL9	V_t
124	P⇄S	
125	ST×7	complete $\|V(x)\|$ calculation
126	RCL7	
127	LOG	
128	2	
129	0	
130	×	
131	PRTX	20 log V(x)
132	RCL9	
133	PRTX	$\angle V(x)$
134	*LBL9	space and return subroutine
135	SPC	
136	RTN	

NOTE FLAG SET STATUS

LABELS

A $\|Z_t\| \uparrow \angle Z_t$	B source voltage	C	D	E calc V(x)
a $ReZ_t \uparrow ImZ_t$	b	c	d	e
0	1 ρ calc jump	2	3	4
5	6	7	8	9 spc & rtn

FLAGS

0	
1	
2	ρ calc ?
3	

SET STATUS

FLAGS (ON OFF)	TRIG	DISP
0	DEG ■	FIX ■
1	GRAD	SCI
2 ■	RAD	ENG
3		n 2

PROGRAM 1-3 SECOND ORDER ACTIVE NETWORK TRANSMISSION FUNCTION.

Program Description and Equations Used

This program provides the coefficients of the numerator and denominator polynomials of the transmission function T(s) = N(s)/D(s), of the generalized second order active network shown in Fig. 1-3.1. A second part of the program provides the polynomial roots. If a real (non-ideal) operational amplifier (op-amp) is used, the amplifier will have both finite gain and bandwidth. The compensation pole of the op-amp will introduce a parasitic pole causing D(s) to become third order even though the RC network is set up to provide second order response. This program accepts the gain and 3 dB bandwidth of the amplifier and calculates the resulting third order transmission function.

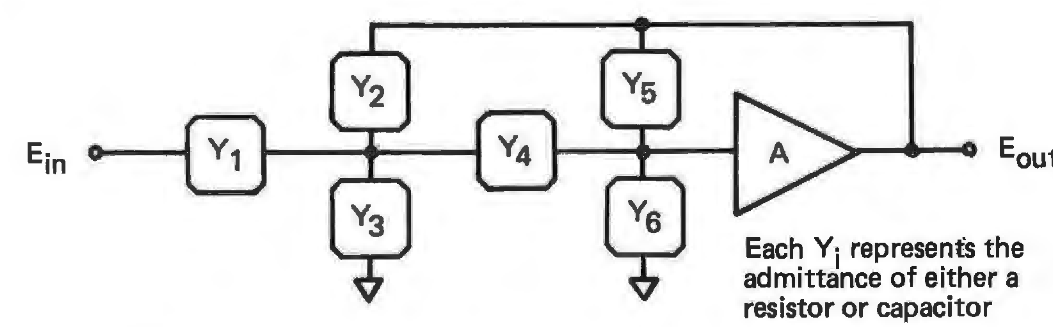


Figure 1-3.1 Generalized second order circuit.

If the natural frequencies of the response governed by the RC network alone are many decades removed from the amplifier unity gain crossover frequency, then the transmission function T(s), will be practically equal to the transmission function of the second order network with an ideal infinite bandwidth amplifier. The component values dictated by many active filter references assume ideal operational amplifier characteristics.

When the natural frequencies are within a decade or two of the amplifier unity gain crossover frequency, then the parasitic pole will cause a noticeable shift in the natural frequencies governed by the RC network alone. The network can be predistorted so the natural frequencies shift to the desired positions (see Program 2-11).

The transmission function is determined by writing the nodal equations for the network, and solving for E_{out} in terms of E_{in}. This derivation is done later and provides:

$$E_{out} = \frac{A_0 Y_1 Y_4}{D(s)} \qquad (1\text{-}3.1)$$

where

$$D(s) = (Y_1 + Y_2 + Y_3)\left[(Y_4 + Y_6)(1 + \tau s) + Y_5\,(1 - A_0 + \tau s)\right] + Y_4\left[\,Y_6 + (1 + \tau s) + Y_5\,(1 - A_0 + \tau s) - A_0 Y_2\right]$$

and where a one-pole model of the amplifier is assumed:

$$A = \frac{A_0}{1 + \tau s} \qquad (1\text{-}3.2)$$

The sign of A_0 may be either positive or negative depending upon the amplifier characteristics (see examples). The first program uses Eq. (1-3.1) to form the numerator and denominator polynomials, and the second program finds the zeros of these polynomials (polynomial roots).

When the element values are loaded, capacitors are signified by a negative mantissa. The subroutine under label 8 tests the sign of the entry; if it is negative, the absolute value is stored; if it is positive, it is a resistor, and the reciprocal is taken to convert to conductance, and then multiplied by 10^{50} before storage.

The magnitude of the stored element value is used to signal whether the element is a resistor or a capacitor. Other programs use the sign of the stored value to differentiate between resistors and capacitors, but that indicator cannot be used in this program because algebraic operations are performed on the element values in the main program before the element type subroutine is entered and the resistor/capacitor test is done, i.e., the term $Y_5(1 - A_0)$ can have either sign depending upon the magnitude and sign of A_0, and Y_5 can legitimately represent the admittance of either a resistor or a capacitor.

The magnitude test is done in the summing routine under label 0.

If the absolute value of the coefficient is greater than 10^{30}, it is assumed to be a conductance (s^0 term), the value is divided by 10^{50} to undo the original storage operation, and the summation is done in the stack. If the absolute value of the coefficient is less than 10^{30}, it is assumed to be a capacitance (s^1 term), and the summation is done in the designated i register.

Some terms in the denominator of Eq. (1-3.1) contain the factor τs. These terms generate s^1 and s^2 coefficients. Subroutine 3 is used to perform multiplication by τs and to append the s^1 and s^2 terms to the presently stored s^0 and s^1 terms to form the complete admittance sum set for the denominator segment being evaluated.

After each set of admittance sums (s^0, s^1, & s^2) are calculated and stored, polynomial multiplication is done to generate the coefficients of the various powers of s in the denominator polynomial. This multiplication is accomplished by the routine under label 6. If flag 0 is set, the polynomial coefficient registers are cleared before multiplication. This condition exists for the first product-of-sums. Flag 0 is cleared for the second product-of-sums to indicate continued summation into the polynomial coefficient registers.

After the denominator has been calculated, the polynomial coefficients are normalized by dividing by the s^0 polynomial coefficient. The numerator coefficient is likewise normalized, and the polynomial coefficients are provided as output. This normalization process can cause the program to halt displaying "ERROR" for certain classes of degenerate networks, e.g., a differentiator constructed with capacitors in locations 1 and 4, no elements in locations 2, 3, and 6, and feedback resistor in location 5. The series capacitors should be combined into a single capacitor in location 1 or 4 with the feedback resistor in location 2 or 5 and no elements in locations 3 and 6. The unspecified series elements can be 1 ohm resistors.

The second program finds the zeros of the denominator polynomial (poles of the transmission function). The numerator polynomial will be either a constant, a single zero at the origin, or a double zero at the origin depending on whether the filter is lowpass, bandpass, or highpass, respectively. The second program also indicates the degree of the zero, and the gain constant of the second order pair, K, after the third order root has been removed (if any), i.e.:

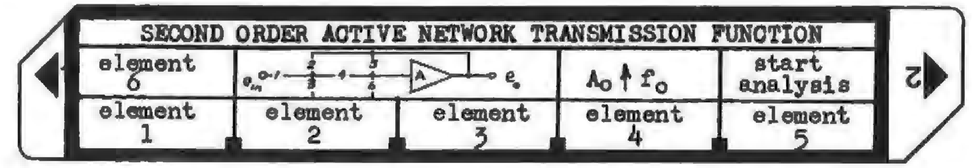
Lowpass: $$T(s) = K \frac{1}{\frac{s^2}{\omega_n^2} + \frac{s}{\omega_n Q} + 1} \quad (1\text{-}3.3)$$

Bandpass: $$T(s) = K \frac{\frac{s}{\omega_n Q}}{\frac{s^2}{\omega_n^2} + \frac{s}{\omega_n Q} + 1} \quad (1\text{-}3.4)$$

Highpass: $$T(s) = K \frac{\frac{s^2}{\omega_n^2}}{\frac{s^2}{\omega_n^2} + \frac{s}{\omega_n Q} + 1} \quad (1\text{-}3.5)$$

If the denominator polynomial is second order, the quadratic formula is used to find the zeros. If it is third order, a Newton-Raphson iterative technique is used to find the real third order zero (there will be at least one), then the third order polynomial is deflated to second order, and the quadratic formula is used to find the remaining zeros of the polynomial. If the zeros of the denominator polynomial are complex, the program will also calculate the natural frequency, $f_n = \omega_n/2\pi$, and the Q, or quality factor of the complex pair (see the equation derivation part of this description for equations and details).

User Instructions



STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Enter element 1			
	a) if resistor (value ≠ 0)	R, ohms	[A]	
	b) if capacitor, enter negative value	C, farad	[chs] [A]	
3	Enter element 2			
	a) if resistor	R, ohms	[B]	
	b) if capacitor	C, farad	[chs] [B]	
	c) if no element present	zero	[B]	
4	Enter element 3			
	a) if resistor	R, ohms	[C]	
	b) if capacitor	C, farad	[chs] [C]	
	c) if no element present	zero	[C]	
5	Enter element 4			
	a) if resistor (value ≠ 0)	R, ohms	[D]	
	b) if capacitor	C, farad	[chs] [D]	
6	Enter element 5			
	a) if resistor	R, ohms	[E]	
	b) if capacitor	C, farad	[chs] [E]	
	c) if no element present	zero	[E]	
7	Enter element 6			
	a) if resistor	R, ohms	[f] [A]	
	b) if capacitor	C, farad	[chs] [f] [A]	
	c) if no element present	zero	[f] [A]	
8	Enter operational amplifier parameters	A_o	[↑]	
		f_o, Hz	[f] [D]	
9	Start analysis		[f] [E]	Den coefs
				Num coefs
10	Go back and change any element then rerun step 9, or load second card to find denominator pole locations, f_n, and Q			

User Instructions

◀1	SECOND AND THIRD ORDER ROOT FINDER PROGRAM USE AFTER TRANSMISSION FUNCTION PROGRAM		START	2▶

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card when display flashes, program execution begins unaided			
2	Program output			
2a	If three real roots, $(s+a)(s+b)(s+c)$			$-a$
				$-b$
				$-c$
2b	If one real root and a complex conjugate pair, $(s+a)(s+\alpha+j\beta)(s+\alpha-j\beta)$			$-a$
				β
				$-\alpha$
				$-\beta$
				$-\alpha$
			of second order pair	f_n (Hz)
				Q
				midband gain
				num zero locations
2c	If two real roots: $(s+a)(s+b)$			$-a$
				$-b$
2c	A complex conjugate pair, $(s+\alpha+j\beta)(s+\alpha-j\beta)$			β
				$-\alpha$
				$-\beta$
				$-\alpha$
			of second order pair	f_n (Hz)
				Q
				midband gain
				num zero locations

User Instructions

TI-59 TRANSLATION

SECOND ORDER ACTIVE NETWORK TRANSMISSION FUNCTION				
init				start analysis
k	R_k	C_k	A_o	f_{-3} dB

TI-59 TRANSLATION

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card one			
2	Initialize and clear registers		2nd A	0
3	Load elements			
	a) load element number (1 to 6)	k	A	k
	b) load element values:			
	if resistor	R_k, ohms	B	R_k
	if capacitor	C_k, F	C	C_k
	if no element present	0	C	0
	Repeat step 3 until all elements have been entered.			
4	Load amplifier dc gain (load negative gain for inverting op-amp)	A_o	D	A_o
5	Load -3 dB rolloff frequency of amplifier	f_{-3} dB, Hz	E	f_{-3} dB
6	Start analysis		2nd E	den coefs
				b_3
			R/S*	b_2
			R/S*	b_1
			R/S*	1
				num coefs
			R/S*	a_2
			R/S*	a_1
			R/S*	a_o
	* "R/S" not necessary if the TI-59 is attached to the PC-100A printer. All results will be printed automatically after the program is started.			

1-3 User Instructions

TI-59 TRANSLATION

SECOND AND THIRD ORDER ROOT FINDER PROGRAM use with 1-3				
				start

TI-59 TRANSLATION

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
7	Load both sides of program card 2			
8	Start second program		[E]	
	a) If three real roots:			-a
	(s+a)(s+b)(s+b)		[R/S*]	-b
			[R/S*]	-c
	b) If one real root and a complex conjugate			-a
	pair: (s+a)(s+α+jβ)(s+α−jβ)			
			[R/S*]	β
			[R/S*]	-α
			[R/S*]	-β
			[R/S*]	-α
			[R/S*]	f_n, Hz
			[R/S*]	Q
			[R/S*]	midband gain
	c) If two real roots: (s+a)(s+b)			-a
			[R/S*]	-b
	d) If a complex conjugate pair:			β
	(s+α+jβ)(s+α−jβ)		[R/S*]	-α
			[R/S*]	-β
			[R/S*]	-α
			[R/S*]	f_n, Hz
			[R/S*]	Q
			[R/S*]	midband gain
	* "R/S" not necessary if the TI-59 is attached to the PC-100A printer. All results will be automatically printed after the program is started.			

Example 1-3.1

The schematic in Fig. 1-3.2 represents a second order active bandpass filter using the infinite gain, multiple feedback topology. The filter element values were designed assuming the op-amp to be ideal, i.e., having infinite gain and bandwidth. The type 741 op-amp is not ideal in that it has both finite gain and bandwidth. This example will use the program to show that the element values provide the desired specification when the op-amp has very large gain (-10^9) and infinite bandwidth ($\tau = 0$). The program will then be run with the gain and bandwidth values for the 741 type op-amp to show that both the pole natural frequency and "Q" have shifted away from the desired values. The 741 has a typical gain of -100,000, and open loop break frequency of 5 Hz.

The design specifications for the filter are:

center frequency:	10 kHz
midband gain:	10
quality factor, Q:	10
capacitor value:	1000 pF

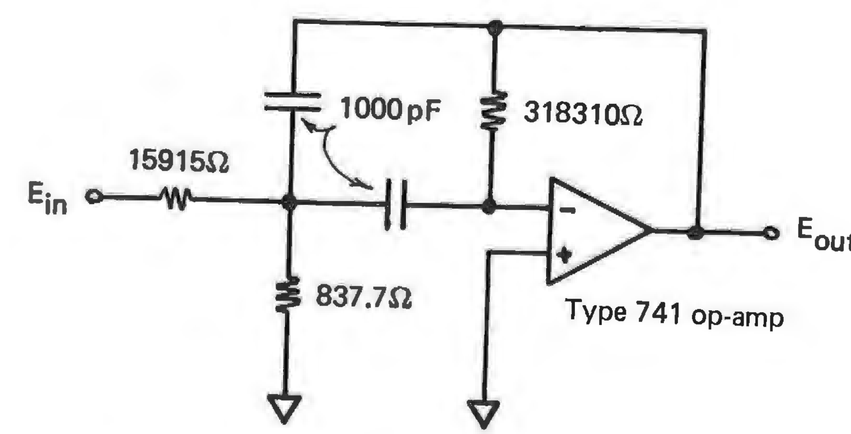


Figure 1-3.2 Second order bandpass active filter, infinite gain-multiple feedback topology.

HP-97 PRINTOUT FOR EXAMPLE 1-3.1

		load first program and enter element values
15915.	GSBA	element 1, resistor
-1.-09	GSBB	element 2, cap
837.7	GSBC	element 3, resistor
-1.-09	GSBD	element 4, cap
318310.	GSBE	element 5, resistor
0.	GSBa	element 6, missing
-1.+09	ST08	enter infinite gain app
0.	ST07	set τ to zero (BW=∞)
	GSBe	start analysis
0.000+00	***	s^3 denominator coef
253.3-12	***	s^2 " "
1.592-06	***	s^1 " "
1.000+00	***	s^0 " "
0.000+00	***	s^2 numerator coef
-15.92-06	***	s^1 " "
0.000+00	***	s^0 " "
		load second card and start analysis
62.75+03	***	imag } complex conjugate poles
-3.142+03	***	real } complex conjugate poles
-62.75+03	***	imag } complex conjugate poles
-3.142+03	***	real } complex conjugate poles
10.00+03	***	f_n } of second order pole pair
10.00+00	***	Q } of second order pole pair
-10.00+00	***	midband gain
0.000+00	***	numerator zero location

		reload first card
-100000.	ENT↑	741 dc gain
5.	GSBd	741 break freq
	GSBe	start analysis
80.63-18	***	s^3
355.1-12	***	s^2
1.913-06	***	s^1
1.000+00	***	s^0
0.000+00	***	s^2
-15.92-06	***	s^1
0.000+00	***	s^0
		load second card & start analysis
-4.400+06	***	real pole location
53.04+03	***	imag } complex conjugate poles
-2.376+03	***	real } complex conjugate poles
-53.04+03	***	imag } complex conjugate poles
-2.376+03	***	real } complex conjugate poles
8.450+03	***	f_n } of second order pole pair
11.17+00	***	Q } of second order pole pair
-9.441+00	***	midband gain
0.000+00	***	numerator zero location

TI-59 PRINTOUT FOR EXAMPLE 1-3.1

		load first program and enter element values
1.		element #
15915.	R	resistor
2.		element #
1. -09	C	capacitor
3.		element #
837.7	R	resistor
4.		element #
1. -09	C	capacitor
5.		element #
318310.	R	resistor
-1. 09	A	amplifier gain (ideal)
1. 25	F	amplifier BW (ideal)
0. 00 00		s^3 den coef
253.31-12		s^2 " "
1.59-06		s^1 " "
1.00 00		s^0 " "
0.00 00		s^2 num coef
-15.92-06		s^1 " "
0.00 00		s^0 " "
		load second card
62.75 03		imag | complex conj. pole pair
-3.14 03		real | complex conj. pole pair
-62.75 03		imag | complex conj. pole pair
-3.14 03		real | complex conj. pole pair
10.00 03		f_n
10.00 00		Q
-10.00 00		midband gain

		reload first card
-100. 03	A	741 dc gain
5. 00	F	741 break freq
80.63-18		s^3 den coef
355.14-12		s^2 " "
1.91-06		s^1 " "
1.00 00		s^0 " "
0.00 00		s^2 num coef
-15.92-06		s^1 " "
0.00 00		s^0 " "
		load second card
-4.3997 06		real pole location
53.03 03		imag | complex conj. pole pair
-2.3760 03		real | complex conj. pole pair
-53.03 03		imag | complex conj. pole pair
-2.3760 03		real | complex conj. pole pair
8.4499 03		f_n
11.17 00		Q
-9.4414 00		midband gain

Example 1-3.2

Figure 1-3.3 is the schematic of a second order highpass filter using the Sallen and Key controlled source topology. An operational amplifier is connected in the voltage follower configuration to provide the unity gain non-inverting buffer amplifier required. The design procedure assumes infinite bandwidth in this buffer, but physical op-amps, such as the 741 type have finite bandwidth (BW). This example will show how this finite bandwidth affects the filter performance. The design specifications are:

natural frequency, f_o:	10000 Hz
quality factor, Q:	$1/\sqrt{2}$ = 0.707
capacitor value, C_1, C_4:	1 nF
asymptotic high frequency gain:	unity

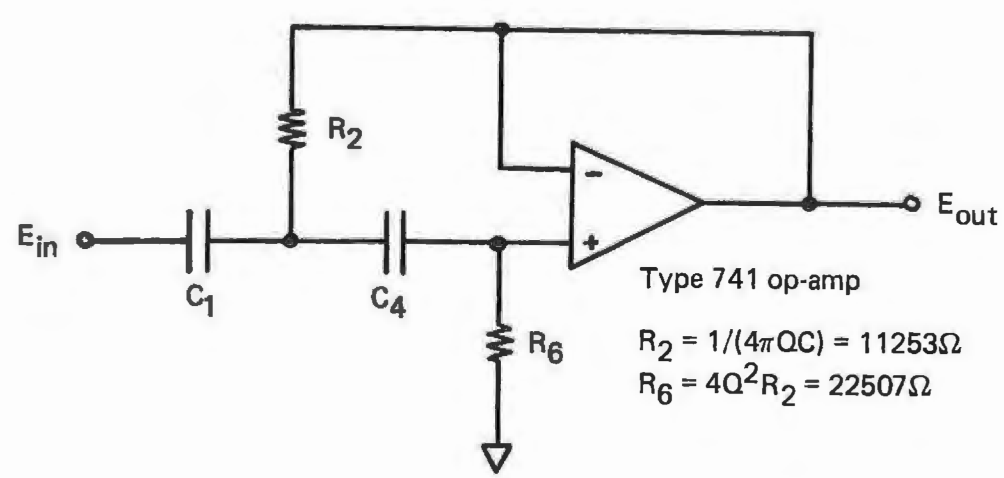


Figure 1-3.3 Sallen and Key type second order highpass filter.

The HP-97 printout is shown on the next page. Again, two runs were made; first the amplifier was assumed to be ideal, and the program output verifies the design specifications; second, the finite gain and bandwidth characteristics of the 741 operational amplifier were used. The program output for the second case shows the non-ideal (finite) characteristics of the 741 have caused the second order pole positions to shift away from the desired positions, and a real pole has also been introduced.

HP-97 PRINTOUT FOR EXAMPLE 1-3.2

load first program and enter element values

-1.-09	GSBA	element 1, capacitor
11253.	GSBB	element 2, resistor
0.	GSBC	element 3, missing
-1.-09	GSBD	element 4, capacitor
0.	GSBE	element 5, missing
22507.	GSBa	element 6, resistor
1.	ST00	set $A_0 = 1$
0.	ST07	set $\tau = 0$ (BW = ∞)
	GSBe	start analysis
0.000+00	***	s^3
253.3-12	***	s^2
22.51-06	***	s^1
1.000+00	***	s^0
253.3-12	***	s^2
0.000+00	***	s^1
0.000+00	***	s^0

load second card & start analysis

44.43+03	***	imag	complex conjugate poles
-44.43+03	***	real	
-44.43+03	***	imag	
-44.43+03	***	real	
10.00+03	***	f_n	of second order pole pair
707.1-03	***	Q	
1.000+00	***	asymptotic gain	
0.000+00	***	numerator zero locations	
0.000+00	***		

Reload first card and enter op-amp parameters

1.	ENT↑	gain	type 741
500000.	GSBd	bandwidth	
	GSBe	start analysis	
80.62-18	***	s^3	denominator coef
267.6-12	***	s^2	"
22.82-06	***	s^1	"
1.000+00	***	s^0	"
253.3-12	***	s^2	numerator coef
0.000+00	***	s^1	"
0.000+00	***	s^0	"

load second card & start analysis

-3.233+06	***	real pole location	
44.40+03	***	imag	complex conjugate poles
-43.19+03	***	real	
-44.40+03	***	imag	
-43.19+03	***	real	
9.858+03	***	f_n	of second order pole pair
717.0-03	***	Q	
971.7-03	***	asymptotic gain	
0.000+00	***	numerator zero locations	
0.000+00	***		

TI-59 PRINTOUT FOR EXAMPLE 1-3.2

load first program and enter element values

1.		element #	
1. -09	C	capacitor	
2.		element #	
11253.	R	resistor	
4.		element #	
1. -09	C	capacitor	
6.		element #	
22507.	R	resistor	
1.	A	amplifier gain (ideal)	
1. 25	F	amplifier BW (ideal)	
0. 00 00		s^3	den coef
253. 27-12		s^2	"
22. 51-06		s^1	"
1. 00 00		s^0	"
253. 27-12		s^2	num coef
0. 00 00		s^1	"
0. 00 00		s^0	"

load second card

44. 43 03	imag	complex conj. pole pair
-44. 43 03	real	
-44. 43 03	imag	
-44. 43 03	real	
10. 00 03	f_n	of second order pole pair
707. 12-03	Q	
1. 00 00	asymptotic gain	

reload first card

1. 00	A	741 gain	
500. 03	F	741 BW	
80. 62-18		s^3	den coef
267. 60-12		s^2	"
22. 82-06		s^1	"
1. 00 00		s^0	"
253. 27-12		s^2	num coef
0. 00 00		s^1	"
0. 00 00		s^0	"

load second card

-3. 23 06	real pole location	
44. 40 03	imag	complex conj. pole pair
-43. 19 03	real	
-44. 40 03	imag	
-43. 19 03	real	
9. 86 03	f_n	of second order pole pair
717. 04-03	Q	
971. 75-03	asymptotic gain	

Derivation of Equations and Algorithms Used

Active network transfer function: The schematic of the generalized second order active network is shown in Fig. 1-3.1. Let the junction of Y_1, Y_2, Y_3, and Y_4 be designated node 1. Furthermore, let the junction of Y_4, Y_5, and Y_6 be designated as node 2. The nodal equations for this circuit may be written in matrix form in terms of the voltages at node 1 (E_1), and at node 2 (E_2):

$$\begin{bmatrix} \{Y_1 + Y_2 + Y_3 + Y_4\} & \{-Y_4\} \\ \{-Y_4\} & \{Y_4 + Y_5 + Y_6\} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ 0 & Y_5 \end{bmatrix} \cdot \begin{bmatrix} E_{in} \\ E_{out} \end{bmatrix} \qquad (1\text{-}3.6)$$

Since $E_2 = E_{out}/A$, this expression is substituted into Eq. (1-3.6), and the dependent variables brought to the left hand side.

$$\begin{bmatrix} \{Y_1 + Y_2 + Y_3 + Y_4\} & \left\{\frac{Y_4}{A} - Y_2\right\} \\ \{-Y_4\} & \left\{\frac{Y_4 + Y_5 + Y_6}{A} - Y_5\right\} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_{out} \end{bmatrix} = \begin{bmatrix} Y_1 \\ 0 \end{bmatrix} (E_{in}) \qquad (1\text{-}3.7)$$

$T(s) = E_{out}/E_{in}$ may be obtained from Eq. (1-3.7) using Cramer's rule. To this end, the determinant of the coefficient matrix (Δ) is needed:

$$\Delta = (Y_1 + Y_2 + Y_3 + Y_4) \cdot \left[\frac{Y_4 + Y_5 + Y_6}{A} - Y_5\right] - Y_4\left[\frac{Y_4}{A} - Y_2\right] \qquad (1\text{-}3.8)$$

After clearing fractions and eliminating term subtraction,

$$A \cdot \Delta = (Y_1 + Y_2 + Y_3)[Y_4 + Y_6 + Y_5(1 - A)] + Y_4\,[Y_5(1 - A) - AY_2 + Y_6] \qquad (1\text{-}3.9)$$

Substituting $A = A_0/(1 + \tau s)$ as the amplifier gain, and clearing fractions, Eq. (1-3.9) becomes:

$$A_0 \cdot \Delta = (Y_1 + Y_2 + Y_3)[(Y_4 + Y_5)(1 + \tau s) + Y_5(1 - A_0 + \tau s)] + Y_4\,[Y_6(1 + \tau s) + Y_5(1 - A_0 + \tau s) - A_0 \cdot Y_2] \qquad (1.3.10)$$

Using Cramer's rule, the transmission function becomes:

$$T(s) = E_{out}/E_{in} = (Y_1 \cdot Y_4)/\Delta \qquad (1\text{-}3.11)$$

Newton-Raphson solution for finding real zeros of third order polynomials: The Newton-Raphson solution is an iterative procedure for finding the values of x where f(x) becomes zero, hence, these values of x are called the zeros of f(x). If the mathematical operations are restricted to real numbers, then the procedure will only find the real zeros of the function, f(x). All odd ordered polynomials with real coefficients have at least one real zero. The third order polynomial generated by this program falls into this class, therefore real arithmetic is used to extract the real zero.

Given the function $f(x) = 0$, the Newton-Raphson solution provides a new estimate, x_{i+1}, based on the present estimate, x_i, and the tangent to $f(x_i)$. The value of x_{i+1} is determined by calculating the intercept of the tangent, $f'(x_i)$ on the x axis as shown in Fig. 1-3.2.

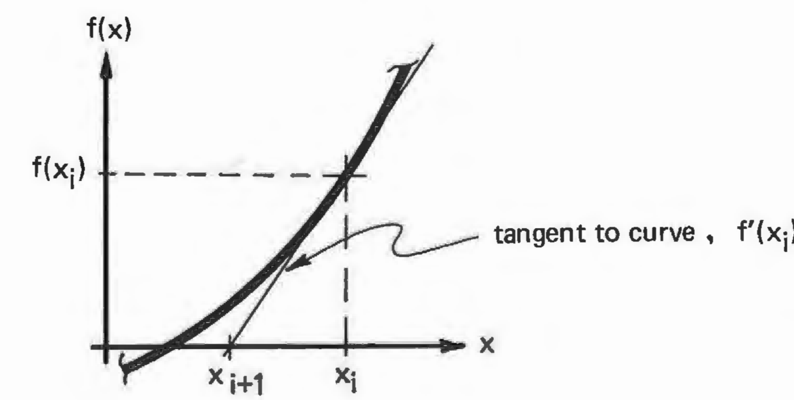


Figure 1-3.2 Newton-Raphson solution method.

$$f'(x_i) = \Delta f(x_i)/\Delta x_i = (f(x_i) - 0)/(x_i - x_{i+1})$$

Solving for x_{i+1}:

$$x_{i+1} = x_i - f(x_i)/f'(x_i)$$

The iteration is stopped when the absolute value of the correction term, $f(x_i)/f'(x_i)$ becomes smaller than the desired error limit, $x_i \cdot 10^{-8}$.

Once the real zero of the third order polynomial has been found, a polynomial division is done to deflate the polynomial to second order. The quadratic equation is used to obtain the zeros of the second order polynomial, and these zeros may be complex. If $s = a$ is a zero of $f(s)$, then $s-a$ must be a factor of $f(s)$, and can be removed:

$$\begin{array}{rl}
 & b_3s^2 + (ab_3 + b_2)s + (a(ab_3 + b_2) + b_1) \\
s-a \big) & b_3s^3 + b_2s^2 + b_1s + 1 \\
 & -(b_3s^3 - ab_3s^2) \\
 & (ab_3 + b_2)s^2 + b_1s + 1 \\
 & -[(ab_3 + b_2)s^2 - a(ab_3 + b_2)s] \\
 & a(ab_3 + b_2)s + b_1s + 1 \\
 & -\{a(ab_3 + b_2)s + b_1s - a[a(ab_3 + b_2) + b_1]\} \\
 & 0^*
\end{array} \tag{1-3.14}$$

The third order polynomial is evaluated in nested form, i.e.:

$$D(s) = (((b_3)s + b_2)s + b_1)s + 1 \tag{1-3.15}$$

When $s = a$, the intermediate products in $D(s)$ are the same as the second order polynomial coefficients in Eq. (1-3.14). These intermediate products are stored at lines 027 and 031 of the program on the second card. The numbers stored only have value in the last iteration loop before loop exit, at which time $s = a$, and $f(s) = 0$, the desired result.

The second order polynomial is normalized so $c_0 = 1$ (lines 064 to 066). This normalization places the second order polynomial in the same form as the third order polynomial was originally. The quadratic formula is now used to find the zeros of the second order polynomial, $c_2s^2 + c_1s + 1$.

$$s_{1,2} = -c_1/(2c_2) \pm \sqrt{(c_1/(2c_2))^2 - 1/c_2} \tag{1-3.16}$$

If the discriminant, $(c_1/(2c_2))^2 - 1/c_2$, is positive, then two real zeros exist, if it is zero, a double zero exists, and if it is negative, a complex conjugate pair of zeros exist. Steps 067 through 102 find the zeros of the second order polynomial.

* By definition since $s = a$ is a zero of the polynomial.

If the zeros of the second order polynomial are complex conjugates, then the poles of the transmission function are also complex conjugates, and a natural frequency, f_n, and quality factor, Q, may be calculated:

$$f_n = 1/(2\pi\sqrt{c_2}) \tag{1-3.17}$$

$$Q = \sqrt{c_2}/c_1 \tag{1-3.18}$$

These calculations are performed by steps 103 through 113 of the program. Assuming the third order real pole of the transmission function (parasitic pole caused by the op-amp characteristics) to be large compared to the other poles, then the gain term, K, can be defined in terms of the numerator and denominator coefficients:

$$T(s) = \frac{a_2s^2 + a_1s + a_0}{(s/a + 1)(c_2s^2 + c_1s + 1)} \tag{1-3.19}$$

$$\text{lowpass case:} \quad K = a_0 \tag{1-3.20}$$

$$\text{bandpass case:} \quad K = a_1/c_1 \tag{1-3.21}$$

$$\text{highpass case:} \quad K = a_2/c_2 \tag{1-3.22}$$

The gain term is calculated by steps 114 through 137 of the program.

1-3 card 1

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD ELEMENT 1
002	EE⁄	
003	GTO8	
004	*LBLB	LOAD ELEMENT 2
005	2	
006	GTO8	
007	*LBLC	LOAD ELEMENT 3
008	3	
009	GTO8	
010	*LBLD	LOAD ELEMENT 4
011	4	
012	GTO8	
013	*LBLE	LOAD ELEMENT 5
014	5	
015	GTO8	
016	*LBLa	LOAD ELEMENT 6
017	6	
018	*LBL8	element load subroutine
019	STOI	store register index
020	R↓	recover and store
021	STO9	element value
022	X>0?	test for resistor
023	GTO8	
024	CHS	
025	GTO7	store capacitor value
026	*LBL8	
027	1/X	calculate conductance,
028	EEX	multiply by 10^{50}, and store
029	5	
030	0	
031	STO8	store 10^{50} for later use
032	x	
033	*LBL7	
034	STOi	store modified element value
035	RCL9	recall original element to
036	RTN	display and return to keybd
037	*LBLd	LOAD A_0 AND f_o OF AMPLIFIER
038	Pi	
039	x	
040	ENT↑	calculate and store
041	+	$\tau = 1/(2\pi f_o)$
042	1/X	
043	STO7	
044	R↓	recover and store A_0
045	STO0	
046	RTN	return control to keyboard
047	*LBLe	START ANALYSIS
048	GSB1	
049	RCL1	calculate s^0 and s^1 terms
050	GSB0	
051	RCL2	of $Y_1 + Y_2 + Y_3$
052	GSB0	
053	RCL3	
054	GSB2	
055	RCL4	
056	GSB0	
057	RCL6	
058	GSB0	
059	RCL5	calculate and store s^0 and
060	EEX	
061	RCL0	s^1 terms of:
062	-	
063	x	$Y_4 + Y_6 + Y_5(1 - A_0)$
064	STO9	
065	GSB2	
066	RCL4	
067	GSB0	calculate and store s^1 and
068	RCL5	
069	GSB0	s^2 terms of:
070	RCL6	
071	GSB0	$\tau s(Y_4 + Y_5 + Y_6)$
072	GSB3	
073	SF0	calculate and store:
074	GSB6	$(Y_1+Y_2+Y_3)\{(Y_4+Y_6)(1+\tau s) + Y_5(1-A_0+\tau s)\}$
075	GSB1	initialize index counter
076	RCL4	calculate and store Y_4
077	GSB2	
078	RCL6	calculate s^0 and s^1 terms
079	GSB0	of:
080	RCL9	
081	GSB0	$Y_6 + Y_5(1 - A_o) - A_oY_2$
082	RCL2	
083	RCL0	
084	x	
085	CHS	
086	GSB2	
087	RCL6	
088	GSB0	calculate and store s^1 and
089	RCL5	
090	GSB0	s^2 terms of $\tau s(Y_5 + Y_6)$
091	GSB3	
092	CF0	clear flag 0 to indicate
093	GSB6	additional summing and
094	P⇄S	calculate and store:
095	RCL0	$Y_4\{Y_6(1+\tau s) + Y_5(1-A_0+\tau s) - A_0 Y_2\}$
096	STOA	normalize denominator terms:
097	ST÷1	
098	ST÷2	$\frac{a_3}{a_0}s^3 + \frac{a_2}{a_0}s^2 + \frac{a_1}{a_0}s + 1$
099	ST÷3	
100	RCL3	recall, store, and print
101	PRTX	normalized denominator terms
102	STOD	
103	RCL2	
104	PRTX	$a_3/a_o \rightarrow R_D$
105	STOC	
106	RCL1	$a_2/a_o \rightarrow R_C$
107	P⇄S	
108	PRTX	$a_1/a_o \rightarrow R_B$
109	STOB	
110	EEX	
111	ENT↑	
112	PRTX	

REGISTERS

0 A_o	1 Y_1	2 Y_2	3 Y_3	4 Y_4	5 Y_5	6 Y_6	7 τ	8 10^{50}	9 scratch
S0 Σs^0 terms	S1 Σs^1 terms	S2 Σs^2 terms	S3 Σs^3 terms	S4	S5 s_1^2	S6 s_1^0	S7 s_1^1	S8 s_2^0	S9 s_2^1

A b_o	B D_1	C D_2	D D_3	E scratchpad	I index

1-3 card 1

Program Listing II

Step	Key	Comment
113	SPC	
114	SF0	indicate summing register clr
115	GSB1	initialize registers and index
116	RCL1	calculate and store Y_1
117	GSB2	
118	RCL4	calculate and store Y_4
119	GSB2	
120	CLX	set s^2 term of Y_4 to zero
121	STOi	
122	GSB6	calculate and store $Y_1 \cdot Y_4$
123	RCL0	
124	P⇄S	normalize numerator terms
125	RCLA	
126	÷	
127	ST×2	
128	ST×1	
129	ST×0	
130	RCL2	
131	PRTX	
132	RCL1	recall and print
133	PRTX	numerator terms
134	RCL0	
135	P⇄S	
136	PRTX	
137	SPC	
138	*LBLb	
139	PSE	wait loop for 2nd card read
140	GTOb	
141	*LBL0	subroutine to determine
142	ENT↑	whether recalled Y element
143	ABS	is a conductance or a
144	EEX	capacitance
145	3	If a conductance, perform
146	0	summation in the stack,
147	X>Y?	otherwise, element is a
148	GTO9	susceptance, and summation is
149	R↓	done in the designated
150	R↓	i register.
151	RCL8	
152	÷	
153	+	
154	RTN	
155	*LBL9	susceptance summation in R(i)
156	R↓	
157	R↓	
158	ST+i	
159	R↓	
160	RTN	
161	*LBL1	register and index
162	1	initialization
163	9	
164	STOI	$19 \rightarrow R_I$
165	CLX	
166	STOi	$0 \rightarrow R_{19}$
167	RTN	
168	*LBL2	finish Y element summation,
169	GSB0	store stack summation, and
170	DSZI	initialize next susceptance
171	STOi	summation register
172	DSZI	
173	CLX	
174	STOi	
175	RTN	
176	*LBL3	multiply by τs to form s^2
177	RCL7	and additional s^1 terms,
178	P⇄S	and add to presently stored
179	ST×5	terms
180	x	
181	ST+7	
182	P⇄S	
183	RTN	
184	*LBL6	polynomial multiplication
185	P⇄S	
186	CLX	
187	GSB5	
188	RCL6	s^0 term calculation
189	RCL8	
190	GSB4	
191	EEX	
192	GSB5	
193	RCL7	
194	RCL8	s^1 term calculation
195	GSB4	
196	RCL6	
197	RCL9	
198	GSB4	
199	2	
200	GSB5	
201	RCL7	
202	RCL9	
203	GSB4	s^2 term calculation
204	RCL5	
205	RCL8	
206	GSB4	
207	3	
208	GSB5	
209	RCL5	
210	RCL9	s^3 term calculation
211	GSB4	
212	P⇄S	
213	RTN	
214	*LBL4	polynomial multiplication
215	x	subroutine
216	ST+i	
217	RTN	
218	*LBL5	initialization subroutine
219	STOI	used with polynomial
220	CLX	multiplication subroutine
221	F0?	
222	STOi	
223	RTN	

LABELS					FLAGS
A load Y_1	B load Y_2	C load Y_3	D load Y_4	E load Y_5	0 continued summation
a load Y_6	b second card wait loop	c	d load A_0 & f_0	e start analysis	1
0 sum R & sC	1 initialize	2 loop terminate	3 calculate s^0 & s^1	4 summation subroutine	2
5 summation initialize	6 polynomial multiplication	7 input routine	8 input routine	9 recall subroutine	3

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	USERS CHOICE	
1			DEG	FIX
2			GRAD	SCI
3			RAD	ENG
				n ___

Program Listing I

```
001  *LBLE   START ANALYSIS
002   SPC
003   P⇄S
004   RCLD   if s^3 coefficient is not
005   X≠0?   zero go to 3rd order soln
006   GTO0   otherwise store remaining
007   RCLC   second order coefficients
008   STO9   and go to second order
009   RCLB   solution
010   STO8
011   GTO2
012  *LBL0   third order solution
013   RCLC
014   X⇄Y    calculate initial guess
015    ÷     for real 3rd order root
016   CHS
017   STO6
018  *LBL1   Newton-Raphson start
019   RCL6
020   ENT↑
021   ENT↑
022   ENT↑
023   RCLD
024    x
025   RCLC
026    +
027   STO8
028    x
029   RCLB   calculate f(x1)
030    +
031   STO7
032    x
033   EEX
034    +
035   STO5
036   CLX
037   RCLD
038    3
039    x
040    x
041   RCLC
042   ENT↑   calculate f'(x1)
043    +
044    +
045    x
046   RCLB
047    +
048   X≠0?   f'(x1) = 0 escape
049   ST÷5   calc f(x1)/f'(x1)
050   RCL5   apply correction to x1
051   ST-6
052   ABS
053   RCL6
054   EEX
055    8
056    ÷     test for loop exit
057   ABS
058   X≤Y?
059   GTO1
060   RCL6   print real root
061   GSB9
062   RCLD
063   STO9   normalize remaining
064   RCL7   second order coefficients
065   ST÷8
066   ST÷9
```

REGISTERS

0	1	2	3	4	5	6	7	8	9
A_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	τ	scratch	scratch
S0 $\sum s_n^0$	S1 $\sum s_n^1$	S2 $\sum s_n^2$	S3	S4	S5 scratch	S6 σ	S7 $c_0, 1/c_2$	S8 $c_1, c_1/2c_2$	S9 c_2

A	B	C	D	E	I
b_0	D_1	D_2	D_3		

Program Listing II

```
067  *LBL2   second order solution
068   RCL9   c2
069   ENT↑
070    +     2*c2
071   ST÷8 }
072   RCL8 } c1/(2c2)
073   X²
074   RCL9
075   1/X
076   STO7
077    -     (c1/(2c2))^2 - 1/c2
078   X<0?   if discriminant is negative,
079   GTO3   go to imaginary solution
080   √X
081   STO5
082   RCL8   calculate and print
083    -     one real root
084   GSB9
085   RCL5
086   RCL8   calculate and print
087    +     other real root
088   CHS
089   GTO0
090  *LBL3   imaginary solution routine
091   CHS
092   √X
093   STO5
094   PRTX   calculate and print
095   RCL8   one imaginary root
096   CHS
097   GSB9
098   RCL5
099   CHS    calculate and print
100   PRTX   other imaginary root
101   X⇄Y
102   GSB9
103   RCL7   ωn^2
104   √X
105    2
106    ÷
107   STO5   ωn/2
108   Pi
109    ÷
110   PRTX   fn, the natural frequency
111   RCL5
112   RCL8
113    ÷     Q, the quality factor
114  *LBL0
115   GSB9   print Q, or second real root
116   RCL9
117   ST×7   restore second order
118   ENT↑   coefficients
119    +
120   ST×8
121   SPC
122   RCL2 } is numerator second order?
123   X≠0? }
124   GTO0
125   RCL0 } is numerator a constant?
126   X≠0? }
127   GTO8
128   RCL1   numerator is first order
129   RCL8   calculate and print the
130    ÷     gain term, K
131   GSB9
132   CLX    print location of numerator
133   GTO8   zero and exit program
134  *LBL0   numerator is second order
135   RCL9
136    ÷     calculate and print the
137   GSB9   gain term, K
138   CLX    print location of the
139   PRTX   numerator zeros
140  *LBL8   program exit, restore
141   P⇄S    registers to original order
142  *LBL9   print and space subroutine
143   PRTX
144   SPC
145   RTN
```

LABELS

A	B	C	D	E start analysis
a	b	c	d	e
0 local label	1 local label	2 2nd order solution	3 imag roots	4
5	6	7	8 P⇄S, prt & space	9 print & space

FLAGS

FLAGS
0
1
2
3

SET STATUS

FLAGS (ON OFF)	TRIG	DISP
0	DEG ■	FIX
1	GRAD	SCI
2	RAD	ENG ■
3		n 3

<u>Suggested program changes for the HP-67:</u> Program space does not allow the inclusion of a print, R/S toggle and associated output routine. To cause the program execution to stop at the data output points, replace the "print" statements with "R/S" statements at the following line numbers: 101, 104, 108, 112, 131, 133, and 136 in program one, and at lines 094, 100, 110, 139, and 143 in program two.

If these changes are made, the program will stop at each output point. To continue program execution, key a "R/S" command from the keyboard.

TI-59 PROGRAM LISTING 1-3 card 1

000	76	LBL	subroutine to sum conductance & susceptance
001	44	SUM	
002	42	STO	store entry in scratchpad
003	09	09	
004	50	IxI	test for conductance: If entry is smaller than 10^{30}, then it is a susceptance and program execution jumps to step 24.
005	32	X:T	
006	01	1	
007	52	EE	
008	03	3	
009	00	0	
010	77	GE	
011	00	00	
012	24	24	
013	43	RCL	recover entry
014	09	09	
015	55	÷	remove conductance scaling
016	01	1	
017	52	EE	
018	05	5	
019	00	0	
020	95	=	
021	74	SM*	sum conductance
022	05	05	
023	92	RTN	return to main program
024	43	RCL	recover entry
025	09	09	
026	74	SM*	sum susceptance
027	04	04	
028	92	RTN	return to main program
029	76	LBL	initialization subroutine
030	59	INT	
031	02	2	initialize susceptance storage register index
032	00	0	
033	42	STO	
034	04	04	
035	02	2	initialize conductance storage register index
036	01	1	
037	42	STO	
038	05	05	
039	61	GTO	jump to step 51 and continue program execution
040	00	00	
041	51	51	
042	76	LBL	subroutine to complete summation
043	85	+	
044	71	SBR	gosub subroutine "sum"
045	44	SUM	
046	02	2	increment storage register indices
047	44	SUM	
048	04	04	
049	44	SUM	

050	05	05	
051	00	0	
052	72	ST*	clear next set of storage registers
053	04	04	
054	72	ST*	
055	05	05	
056	92	RTN	return to main program
057	76	LBL	LOAD ELEMENT INDEX
058	11	A	
059	98	ADV	space paper in printer
060	22	INV	set fix 0 format
061	52	EE	
062	22	INV	
063	57	ENG	
064	99	PRT	print element index
065	85	+	
066	32	X:T	save index entry
067	01	1	
068	00	0	calculate storage register location
069	95	=	
070	42	STO	
071	04	04	
072	32	X:T	recover index to display
073	91	R/S	stop program execution
074	76	LBL	LOAD RESISTOR VALUE
075	12	B	
076	35	1/X	form conductance
077	65	×	
078	32	X:T	save conductance
079	01	1	
080	52	EE	multiply conductance by 10^{50} and indirectly store
081	05	5	
082	00	0	
083	95	=	
084	72	ST*	
085	04	04	
086	03	3	setup to print "R" as annotation on right hand edge of printout
087	05	5	
088	69	OP	
089	04	04	
090	32	X:T	
091	35	1/X	recover resistor entry and print annotated value
092	22	INV	
093	52	EE	
094	69	OP	
095	06	06	
096	91	R/S	stop program execution
097	76	LBL	LOAD CAPACITOR VALUE
098	13	C	
099	72	ST*	

Note: This translation was provided by Mr. Roger Junk.

TI - 59 PROGRAM LISTING 1-3 card 1

100	04	04	indirectly store cap
101	32	X⇌T	save entry
102	01	1	
103	05	5	setup to print "C" on
104	69	OP	right hand edge of paper
105	04	04	
106	32	X⇌T	
107	57	ENG	print capacitor value in
108	69	OP	engineering format along
109	06	06	with annotation
110	91	R/S	stop program execution
111	76	LBL	LOAD OP-AMP DC GAIN, A_o
112	14	D	
113	42	STO	store A_o
114	10	10	
115	32	X⇌T	save entry
116	01	1	
117	03	3	setup to print "A" on
118	69	OP	right hand edge of paper
119	04	04	
120	32	X⇌T	recover entry,
121	98	ADV	space paper, and
122	69	OP	print entry and notation
123	06	06	
124	91	R/S	stop program execution
125	76	LBL	LOAD OP-AMP BREAK
126	15	E	FREQUENCY (-3 dB point)
127	65	×	
128	32	X⇌T	save entry
129	02	2	
130	65	×	
131	89	π	form and store:
132	95	=	$\tau = \frac{1}{2\pi f_{-3\ dB}}$
133	35	1/X	
134	42	STO	
135	17	17	
136	01	1	
137	52	EE	if entry is larger than
138	02	2	10^{20} set τ to zero
139	00	0	
140	77	GE	
141	01	01	
142	46	46	
143	00	0	
144	42	STO	
145	17	17	
146	02	2	
147	01	1	setup to print "F" on
148	69	OP	right hand edge of paper
149	04	04	

150	32	X⇌T	recover entry
151	98	ADV	space paper and
152	69	OP	print annotated entry
153	06	06	
154	91	R/S	stop program execution
155	76	LBL	INITIALIZE
156	16	A'	
157	00	0	
158	42	STO	zero elements 2, 3, 5, & 6
159	12	12	
160	42	STO	
161	13	13	
162	42	STO	
163	15	15	
164	42	STO	
165	16	16	
166	91	R/S	stop program execution
167	76	LBL	START ANALYSIS
168	10	E'	
169	71	SBR	test for printer attached
170	04	04	to calculator
171	75	75	
172	71	SBR	initialize counters and
173	59	INT	registers
174	43	RCL	
175	11	11	
176	71	SBR	
177	44	SUM	
178	43	RCL	calculate and store
179	12	12	s^0 and s^1 terms of:
180	71	SBR	
181	44	SUM	$Y_1 + Y_2 + Y_3$
182	43	RCL	
183	13	13	
184	71	SBR	
185	85	+	
186	43	RCL	
187	14	14	
188	71	SBR	
189	44	SUM	calculate and store
190	43	RCL	s^0 and s^1 terms of:
191	16	16	
192	71	SBR	$Y_4 + Y_6 + Y_5(1 - A_0)$
193	44	SUM	
194	01	1	
195	75	-	
196	43	RCL	
197	10	10	
198	95	=	
199	65	×	

TI - 59 PROGRAM LISTING 1-3 card 1

200	43	RCL	
201	15	15	
202	95	=	
203	42	STO	
204	19	19	
205	71	SBR	
206	85	+	
207	43	RCL	
208	14	14	
209	71	SBR	
210	44	SUM	calculate and store
211	43	RCL	s^1 and s^2 terms of:
212	15	15	
213	71	SBR	$\tau s(Y_4 + Y_5 + Y_6)$
214	44	SUM	
215	43	RCL	
216	16	16	
217	71	SBR	
218	44	SUM	
219	71	SBR	
220	49	PRD	
221	86	STF	calculate and store:
222	00	00	$(Y_1+Y_2+Y_3)\{(Y_4+Y_6)(1+\tau s)+Y_5(1-A_o+\tau s)\}$
223	71	SBR	$= D_1$
224	65	×	
225	71	SBR	
226	59	INT	initialize indices
227	43	RCL	calculate and store
228	14	14	s^0 and s^1 terms of
229	71	SBR	Y_4
230	85	+	
231	43	RCL	
232	16	16	
233	71	SBR	
234	44	SUM	calculate and store
235	43	RCL	s^0 and s^1 terms of:
236	19	19	
237	71	SBR	$Y_6 + Y_5(1 - A_o) + A_oY_2$
238	44	SUM	
239	43	RCL	
240	12	12	
241	65	×	
242	43	RCL	
243	10	10	
244	95	=	
245	94	+/-	
246	71	SBR	
247	85	+	
248	43	RCL	
249	16	16	

250	71	SBR	
251	44	SUM	
252	43	RCL	calculate and store
253	15	15	s^1 and s^2 terms of:
254	71	SBR	
255	44	SUM	$\tau s(Y_5 + Y_6)$
256	71	SBR	
257	49	PRD	
258	22	INV	calculate and store:
259	86	STF	
260	00	00	$D_1 + Y_4\{Y_6(1+\tau s) +$
261	71	SBR	$Y_5(1-A_o+\tau s) - A_oY_2\}$
262	65	×	
263	29	CP	
264	43	RCL	test for non-zero
265	00	00	denominator coefs
266	22	INV	
267	67	EQ	
268	02	02	
269	78	78	
270	43	RCL	non-zero test continued
271	01	01	
272	22	INV	
273	67	EQ	
274	02	02	
275	78	78	
276	43	RCL	non-zero test concluded
277	02	02	
278	42	STO	
279	18	18	
280	35	1/X	normalize denominator
281	49	PRD	terms
282	00	00	
283	49	PRD	
284	01	01	
285	49	PRD	
286	02	02	
287	49	PRD	
288	03	03	
289	43	RCL	
290	03	03	recall and print s^3
291	71	SBR	denominator coefficient
292	98	ADV	(program will stop if
293	42	STO	printer is not attached)
294	29	29	
295	43	RCL	
296	02	02	recall and print s^2
297	71	SBR	denominator coefficient
298	04	04	
299	64	64	

TI-59 PROGRAM LISTING 1-3 card 1

300	42	STO	
301	28	28	
302	43	RCL	recall and print s^1 denominator coefficient
303	01	01	
304	71	SBR	
305	99	PRT	
306	42	STO	
307	27	27	
308	43	RCL	recall and print s^0 denominator coefficient
309	00	00	
310	71	SBR	
311	99	PRT	
312	42	STO	
313	26	26	
314	86	STF	indicate first product of sums
315	00	00	
316	71	SBR	initialize indices
317	59	INT	
318	43	RCL	calculate and store s^0, s^1, and s^2 terms of $Y_1 \cdot Y_4$
319	11	11	
320	71	SBR	
321	85	+	
322	43	RCL	
323	14	14	
324	71	SBR	
325	85	+	
326	00	0	
327	72	ST*	
328	04	04	
329	71	SBR	
330	65	×	
331	43	RCL	normalize numerator coefficients
332	10	10	
333	55	÷	
334	43	RCL	
335	18	18	
336	95	=	
337	49	PRD	
338	02	02	
339	49	PRD	
340	01	01	
341	49	PRD	
342	00	00	
343	43	RCL	recall and print s^2 numerator coefficient
344	02	02	
345	71	SBR	
346	98	ADV	
347	43	RCL	recall and print s^1 numerator coefficient
348	01	01	
349	71	SBR	
350	04	04	
351	64	64	
352	43	RCL	recall and print s^0 numerator coefficient
353	00	00	
354	71	SBR	
355	04	04	
356	64	64	
357	91	R/S	stop program execution
358	00	0	unused program memory
359	00	0	
360	00	0	
361	00	0	
362	00	0	
363	00	0	
364	00	0	
365	00	0	
366	00	0	
367	00	0	
368	76	LBL	subroutine to multiply by γs to form s^2 and additional s^1 terms, and add to presently stored terms
369	49	PRD	
370	43	RCL	
371	17	17	
372	49	PRD	
373	24	24	
374	65	×	
375	43	RCL	
376	25	25	
377	95	=	
378	44	SUM	
379	22	22	
380	92	RTN	
381	76	LBL	polynomial multiplication subroutine
382	65	×	
383	00	0	s^0 term calculation
384	71	SBR	
385	04	04	
386	48	48	
387	43	RCL	
388	23	23	
389	65	×	
390	43	RCL	
391	21	21	
392	95	=	
393	74	SM*	
394	05	05	
395	01	1	s^1 term calculation
396	71	SBR	
397	04	04	
398	48	48	
399	43	RCL	

TI-59 PROGRAM LISTING 1-3 card 1

400	22	22	s^1 term calculation continued
401	65	×	
402	43	RCL	
403	21	21	
404	95	=	
405	74	SM*	
406	05	05	
407	43	RCL	
408	23	23	
409	65	×	
410	43	RCL	
411	20	20	
412	95	=	
413	74	SM*	
414	05	05	
415	02	2	s^2 term calculation
416	71	SBR	
417	04	04	
418	48	48	
419	43	RCL	
420	22	22	
421	65	×	
422	43	RCL	
423	20	20	
424	95	=	
425	74	SM*	
426	05	05	
427	43	RCL	
428	24	24	
429	65	×	
430	43	RCL	
431	21	21	
432	95	=	
433	74	SM*	
434	05	05	
435	03	3	s^3 term calculation
436	71	SBR	
437	04	04	
438	48	48	
439	43	RCL	
440	24	24	
441	65	×	
442	43	RCL	
443	20	20	
444	95	=	
445	74	SM*	
446	05	05	
447	92	RTN	
448	42	STO	polynomial multiplication storage subroutine
449	05	05	
450	22	INV	
451	87	IFF	
452	00	00	
453	04	04	
454	58	58	
455	00	0	
456	72	ST*	
457	05	05	
458	92	RTN	
459	76	LBL	subroutine to print and continue if calculator attached to PC-100A printer, or else to stop program execution and display answer
460	98	ADV	
461	98	ADV	
462	76	LBL	
463	99	PRT	
464	57	ENG	
465	99	PRT	
466	22	INV	
467	87	IFF	
468	01	01	
469	04	04	
470	74	74	
471	91	R/S	
472	22	INV	
473	57	ENG	
474	92	RTN	
475	69	OP	subroutine to sense PC-100A printer is attached to calculator
476	08	08	
477	86	STF	
478	01	01	
479	92	RTN	

REGISTER ALLOCATIONS FOR TI-59 1-3 card 1

register number	contents
0	sum of s^0 terms
1	sum of s^1 terms
2	sum of s^2 terms
3	sum of s^3 terms
4	indirect storage register index
5	indirect storage register index
6	
7	
8	
9	
10	A_0, the op-amp dc gain
11	Y_1
12	Y_2
13	Y_3
14	Y_4
15	Y_5
16	Y_6
17	τ
18	b_0
19	$Y_5(1-A_0)$
20	---------$s_2{}^1$
21	------------$s_2{}^0$
22	---------------$s_1{}^1$
23	------------------$s_1{}^0$
24	---------------------$s_1{}^2$
25	
26	D_0
27	D_1
28	D_2
29	D_3
30	

TI-59 PROGRAM LISTING 1-3 card 2

step	code	key	comment
000	76	LBL	START
001	15	E	
002	71	SBR	test for PC-100A printer attached to calculator
003	04	04	
004	75	75	
005	43	RCL	if s^3 coefficient is zero, go to second order solution routine
006	29	29	
007	29	CP	
008	67	EQ	
009	01	01	
010	19	19	
011	55	÷	calculate initial guess for real third order root $x_0 = D_2/D_3$
012	43	RCL	
013	28	28	
014	95	=	
015	35	1/X	
016	94	+/-	
017	42	STO	
018	23	23	
019	43	RCL	Newton-Raphson start
020	23	23	calculate $f(x_i)$
021	65	×	
022	43	RCL	
023	29	29	
024	85	+	
025	43	RCL	
026	28	28	
027	95	=	
028	42	STO	
029	21	21	
030	65	×	
031	43	RCL	
032	23	23	
033	85	+	
034	43	RCL	
035	27	27	
036	95	=	
037	42	STO	
038	22	22	
039	65	×	
040	43	RCL	
041	23	23	
042	85	+	
043	43	RCL	
044	26	26	
045	95	=	
046	42	STO	
047	25	25	
048	03	3	
049	65	×	

step	code	key	comment
050	43	RCL	calculate $f'(x_i)$
051	29	29	
052	65	×	
053	43	RCL	
054	23	23	
055	85	+	
056	02	2	
057	65	×	
058	43	RCL	
059	28	28	
060	95	=	
061	65	×	
062	43	RCL	
063	23	23	
064	85	+	
065	43	RCL	
066	27	27	
067	95	=	
068	29	CP	$f'(x_i) = 0$ escape
069	67	EQ	
070	00	00	
071	75	75	
072	35	1/X	calc $f(x_i)/f'(x_i)$
073	49	PRD	
074	25	25	
075	43	RCL	apply correction to x_i
076	25	25	
077	94	+/-	
078	44	SUM	
079	23	23	
080	50	I×I	test for loop exit
081	32	X⇌T	
082	43	RCL	
083	23	23	
084	55	÷	
085	01	1	
086	52	EE	
087	08	8	
088	95	=	
089	50	I×I	
090	22	INV	
091	77	GE	
092	00	00	
093	19	19	
094	43	RCL	print real root
095	23	23	
096	71	SBR	
097	04	04	
098	65	65	
099	43	RCL	

TI-59 PROGRAM LISTING 1-3 card 2

100	29	29	
101	42	STO	
102	20	20	
103	43	RCL	
104	22	22	
105	29	CP	
106	67	EQ	normalize second order
107	01	01	coefficients
108	31	31	
109	35	1/X	
110	49	PRD	
111	20	20	
112	49	PRD	
113	21	21	
114	49	PRD	
115	22	22	
116	61	GTO	
117	01	01	
118	31	31	
119	43	RCL	
120	28	28	
121	42	STO	store second order
122	20	20	coefficients
123	43	RCL	
124	27	27	
125	42	STO	
126	21	21	
127	43	RCL	
128	26	26	
129	42	STO	
130	22	22	
131	43	RCL	second order solution:
132	20	20	
133	35	1/X	
134	49	PRD	
135	22	22	
136	55	÷	
137	02	2	
138	95	=	calculate discriminant,
139	49	PRD	
140	21	21	$b^2 - 4ac$
141	43	RCL	
142	21	21	
143	33	X²	
144	75	-	
145	43	RCL	
146	22	22	
147	95	=	
148	29	CP	test for negative
149	22	INV	discriminant

150	77	GE	
151	01	01	
152	75	75	
153	34	√X	
154	42	STO	
155	25	25	
156	75	-	
157	43	RCL	calculate and print
158	21	21	one real root
159	95	=	
160	71	SBR	
161	04	04	
162	65	65	
163	43	RCL	
164	25	25	
165	85	+	
166	43	RCL	
167	08	08	calculate and print
168	94	+/-	other real root
169	71	SBR	
170	04	04	
171	65	65	
172	61	GTO	
173	02	02	
174	25	25	
175	94	+/-	imaginary solution:
176	34	√X	
177	42	STO	
178	25	25	
179	71	SBR	
180	04	04	
181	65	65	calculate and print
182	43	RCL	one complex root
183	21	21	
184	94	+/-	
185	71	SBR	
186	04	04	
187	66	66	
188	43	RCL	
189	25	25	
190	94	+/-	
191	71	SBR	
192	04	04	
193	65	65	
194	43	RCL	calculate and print
195	21	21	the other complex root
196	94	+/-	
197	71	SBR	
198	04	04	
199	66	66	

TI-59 PROGRAM LISTING 1-3 card 2

200	68	NOP	
201	68	NOP	
202	43	RCL	
203	22	22	
204	34	√X	
205	55	÷	
206	02	2	
207	95	=	
208	42	STO	calculate and print
209	25	25	f_n, the natural
210	55	÷	frequency
211	89	π	
212	95	=	
213	71	SBR	
214	04	04	
215	65	65	
216	43	RCL	
217	25	25	
218	55	÷	
219	43	RCL	
220	21	21	calculate and print Q
221	95	=	
222	71	SBR	
223	04	04	
224	66	66	
225	43	RCL	
226	20	20	
227	49	PRD	
228	22	22	restore second order
229	65	×	coefficients
230	02	2	
231	95	=	
232	49	PRD	
233	21	21	
234	43	RCL	
235	02	02	
236	22	INV	
237	67	EQ	is numerator second order
238	02	02	
239	55	55	
240	43	RCL	
241	00	00	
242	22	INV	
243	67	EQ	is numerator a constant?
244	02	02	
245	59	59	
246	43	RCL	
247	01	01	
248	55	÷	
249	43	RCL	

250	21	21	
251	95	=	
252	61	GTO	
253	02	02	
254	59	59	
255	55	÷	
256	43	RCL	calculate and print
257	20	20	the gain term, K
258	95	=	
259	71	SBR	
260	04	04	
261	65	65	
262	61	GTO	
263	91	R/S	
264	00	0	
265	00	0	
266	00	0	
267	00	0	
268	00	0	
269	00	0	
270	00	0	
457	00	0	
458	00	0	
459	00	0	
460	76	LBL	subroutine to lock up
461	91	R/S	"R/S" command-prevents
462	61	GTO	further program execution
463	04	04	via the "R/S" command
464	61	61	
465	98	ADV	
466	99	PRT	
467	68	NOP	print or display
468	22	INV	subroutine
469	87	IFF	
470	01	01	
471	04	04	
472	74	74	
473	91	R/S	
474	92	RTN	
475	69	OP	
476	08	08	PC-100A sense routine
477	86	STF	
478	01	01	
479	92	RTN	

REGISTER ALLOCATIONS FOR TI-59 1-3 card 2

register number	contents
0	N_0
1	N_1
2	N_2
3	
4	
5	
6	
7	
8	
9	
10	A_0
11	Y_1
12	Y_2
13	Y_3
14	Y_4
15	Y_5
16	Y_6
17	τ
18	b_0
19	
20	c_2
21	c_1, $c_1/2c_2$
22	c_0, $1/c_2$
23	x_i
24	
25	$f(x_i)/f'(x_i)$
26	D_0
27	D_1
28	D_2
29	D_3

PROGRAM 1-4 L N A P, LADDER NETWORK ANALYSIS PROGRAM.

Program Description and Equations Used

This program evaluates the frequency response and input impedance of a RLC ladder network containing up to 4 nodes and 8 branches using a sweep of discrete evaluation frequencies. The frequency response is provided as magnitude (dB) and phase (degrees, radians, or grads), and the input impedance is provided as real and imaginary parts (ohms). The evaluation frequency may be incremented in a linear manner using an additive increment or in a logarithmic manner using a multiplicative increment.

Each branch of the ladder may contain a resistor (R), a capacitor (C), an inductor (L), a series RC, a parallel RC, a series RL, or a parallel RL network. All element values are stored, and may be reviewed at any time to check or correct the component values and interconnection.

Because of the number of available storage registers in the HP-67/97, the number of nodes cannot exceed four, while the TI-59 can accommodate the data for ten nodes. Elements that inhibit signal flow through the network are not allowed, and will cause the program execution to halt displaying "Error." Examples of these inhibiting elements are a shunt resistor or a shunt inductor having zero value, or series capacitors in series branches having zero values.

The algorithm used by this program assumes 1 volt at the output of the ladder network (see Fig. 1-4.1). From the knowledge of the last branch admittance, the complex branch current may be determined. Since no current flows out of the last node, the last shunt branch current must also flow through the preceding series branch. The complex voltage drop across this branch may be determined by multiplying the branch impedance and the branch current. By adding the series branch voltage to the last node voltage, the next lower node voltage may be obtained. This node voltage times the shunt node admittance will yield the shunt node current. Adding this shunt node current to the previous series branch current will yield the next lower series branch current.

This loop is repeated until the input voltage source is reached (node 0). The frequency response is found from Eq. (1-4.1) and the input impedance from Eq. (1-4.2), i.e.:

$$T(j\omega) = E_{out}/E_{in} = 1/E_0 \tag{1-4.1}$$

$$Z_{in}(j\omega) = E_0/I_0 \tag{1-4.2}$$

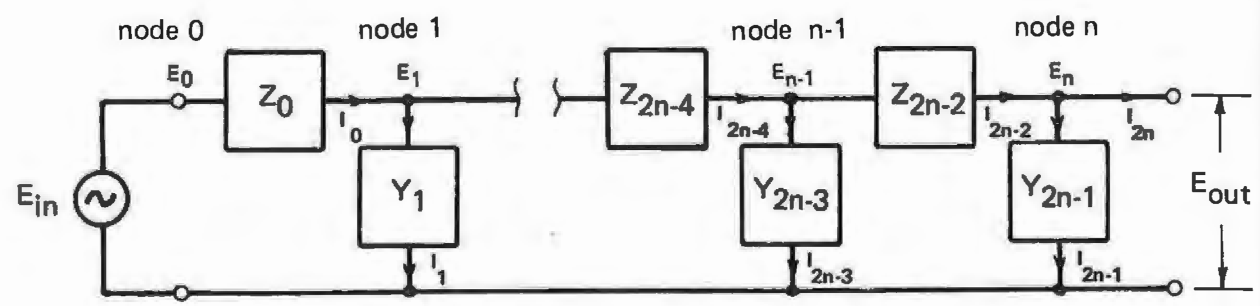


Figure 1-4.1 General ladder network topology.

The preceding algorithm may be expressed in mathematical shorthand using indices:

$$I_{2k-2} = (E_k)(Y_{2k-1}) + I_{2k} \tag{1-4.3}$$

$$E_{k-1} = (I_{2k-2})(Z_{2k-2}) + E_k \tag{1-4.5}$$

where k = n, n-1, n-2, ... , 1, and n is the highest numbered node. The initial conditions for the n-th node are given by:

$$I_{2n} = 0 \tag{1-4.5}$$

$$E_n = 1 + j0 \tag{1-4.6}$$

Equation (1-4.3) is evaluated for I_{2k-2} and substituted into Eq. (1-4.4) to obtain the next lower numbered node voltage. The index, k, is decremented by one, and Eqs. (1-4.3) and (1-4.4) are again evaluated. This process is continued until the voltage at node 0 is obtained. Equation (1-4.1) is used to find the frequency response, $T(j\omega)$, from the node 0 voltage, and Eq. (1-4.2) is used to find the input impedance.

1-4 User Instructions

L N A P, LADDER NETWORK ANALYSIS PROGRAM				
# of nodes	linear sweep	log sweep	data review	start analysis
load R	load L	load C	load start freq	load freq increment

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of the magnetic card			
2	Load the number of nodes in the network			
	The number of nodes cannot exceed four	# nodes	[f] [A]	# branches
3	Enter data starting with the highest numbered node:			
	a) If parallel RC or RL:			
	key in resistance and change sign*	R, ohms	[chs*] [A]	branch #
	key in inductance	L, henry	[B]	br # - 1
	OR key in capacitance	C, farad	[C]	br # - 1
	b) If series RC or RL:			
	key in resistance	R, ohms	[A]	branch #
	OR key in inductance	L, henry	[B]	br # - 1
	key in capacitance	C, farad	[C]	br # - 1
	For resistance, inductance, or capacitance alone in one branch, use step 3b with zero resistance for L or C entry, or use zero inductance for resistance entry. A zero or positive resistance is interpreted as a series branch indication.			
	Alternately, step 3a may be used to enter single inductors or capacitors by entering a very large negative resistance like -10^{20} ohms.			
	Fastest program execution will result if the zero resistance method with step 3b is used for series branches, and the large negative resistance method with step 3a is used for shunt branches. By observing this convention, the program will not use the series to parallel conversion subroutine which requires about 2 seconds to execute each time it is called.			
	Repeat step 3 until all branches including branch 0 have been entered.			
	* The sign change must affect the mantissa and not the exponent on numbers entered using scientific notation.			

User Instructions

LNAP

CONTINUED

2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
4	To review stored element values		[f] [D]	branch #
				R_n*
	* A negative resistance value indicates			L_n or C_n**
	a parallel connection of elements			space
				R_{n-1}*
	** A negative value for the reactive			L C **
	element indicates the element is a			⋮
	capacitor. The capacitance value is			⋮
	the absolute value of the number given.			R_0*
				L_0 or C_0**
5	To change the value of a stored element:			
	a) Key in branch number to be changed	branch #	[STO] [I]	
	b) Key in correct resistance	R	[A]	
	c) Key in correct reactive element value	L OR	[B] OR	
		C	[C]	
	Repeat step 4 or 5 if desired.			
6	To run analysis:			
	a) Load start frequency in hertz	f_{start}	[D]	
	b) Load frequency increment	f_{incr}	[E]	
	(for linear sweep, the new frequency			
	will be the old frequency plus the			
	increment, and for log sweep, the new			
	frequency will be the old frequency			
	times the increment)			
	c) Select linear or logarithmic sweep			
	For linear sweep		[f] [B]	"0"
	For logarithmic sweep		[f] [C]	"1"
	Steps 6a, 6b, and 6c may be executed in			
	any order.			
	d) Start analysis run		[f] [E]	see Ex.
				(1-4.1)

User Instructions

LNAP

CONTINUED

2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
7	To stop the analysis:			
	Wait until the pause that occurs after the			
	imaginary Z_{in} is printed, then key R/S		[R/S]	
	If the program execution is halted without			
	waiting for the pause, the primary and			
	secondary registers may be left interchanged.			
	If a register interchange is suspected,			
	recall register 0 and check to see that			
	branch 0 resistance is stored there.			
	If the branch 0 reactive element is found in			
	register 0, an interchange has occurred,			
	and a P≷S operation is required.		[P≷S]	

Example 1-4.1

Figure 1-4.2 is the schematic of a predistorted 8th order Butterworth lowpass filter with a -3 dB cutoff frequency of 1000 Hz, and a design impedance level of 1000 ohms. Determine the frequency response and input impedance of this filter over a frequency range of 100 Hz to 10 kHz using a logarithmic sweep with 10 points per decade.

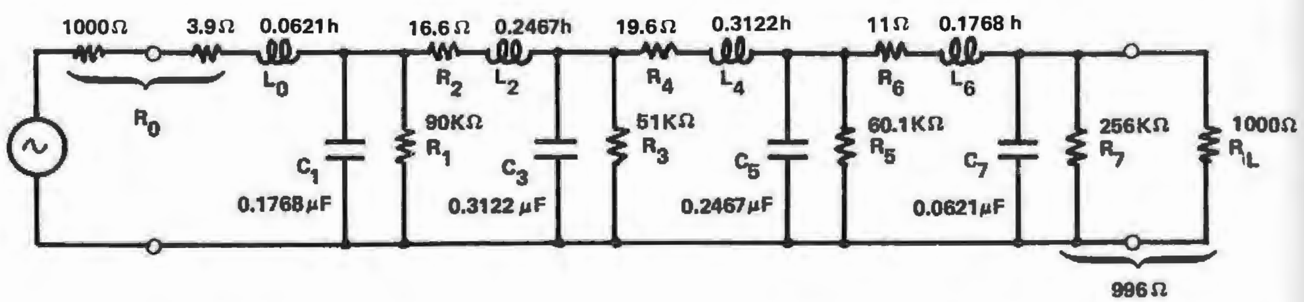


Figure 1-4.2 Predistorted 8th order Butterworth LP filter.

HP-97 PRINTOUT FOR EXAMPLE 1-4.1

PROGRAM INPUT		
4.00	GSBa	load # of nodes
-996.	GSBA	R_7*
.0621-06	GSBC	C_7
11.	GSBA	R_6
.1768	GSBB	L_6
-60100.	GSBA	R_5
.2647-06	GSBC	C_5
19.6	GSBA	R_4
.3122	GSBB	L_4
-51000.	GSBA	R_3
.3122-06	GSBC	C_3
16.6	GSBA	R_2
.2647	GSBB	L_2
-90000.	GSBA	R_1
.1768-06	GSBC	C_1
1003.5	GSBA	R_0
.0621	GSBB	C_0

DATA REVIEW		
	GSBd	start data review
7.000+00	***	branch number
-996.0+00	***	resistive part*
-62.10-09	***	reactive part**
6.000+00	***	
11.00+00	***	
176.8-03	***	
5.000+00	***	
-60.10+03	***	
-264.7-09	***	
4.000+00	***	
19.60+00	***	
312.2-03	***	
3.000+00	***	
-51.00+03	***	
-312.2-09	***	
2.000+00	***	
16.60+00	***	
264.7-03	***	
1.000+00	***	
-90.00+03	***	
-176.8-09	***	
0.000+00	***	
1.004+03	***	
62.10-03	***	

* A negative sign indicates a parallel connection of elements.

** A negative sign indicates a capacitor as the reactive element.

HP-97 PRINTOUT FOR EXAMPLE 1-4.1

```
           GSBc   select log sweep
   100.    GSBD   load start frequency

     .1    10^x   calculate freq  increment for 10 points per decade
1.259+00   ***    multiplicative increment (manual print command)
           GSBE   load multiplicative increment

           GSBe   start analysis
```

PROGRAM OUTPUT

```
 100.0+00 freq, Hz    316.2+00      1.000+03      3.162+03
-6.467+00 gain, dB   -6.482+00     -9.816+00     -86.07+00
-29.40+00 phase, °   -94.00+00      2.263+00      95.46+00
 2.000+03 Re Zin,Ω    2.000+03      5.622+03      1.005+03
 208.2-03 Im Zin,Ω    115.0-03     -415.8+00      932.4+00

 125.9+00             398.1+00      1.259+03      3.981+03
-6.468+00            -6.493+00     -22.54+00     -102.0+00
-37.04+00            -119.2+00     -98.07+00      75.48+00
 2.000+03             2.000+03      1.049+03      1.005+03
 250.4-03            -723.8-03     -813.0+00      1.319+03

 158.5+00             501.2+00      1.585+03      5.012+03
-6.470+00            -6.513+00     -38.24+00     -118.0+00
-46.68+00            -152.0+00     -161.3+00      59.79+00
 2.000+03             1.993+03      1.013+03      1.004+03
 292.3-03            -2.388+00     -139.0+00      1.772+03

 199.5+00             631.0+00      1.995+03      6.310+03
-6.472+00            -6.557+00     -54.14+00     -134.0+00
-58.87+00             164.2+00      154.3+00      47.41+00
 2.000+03             1.957+03      1.008+03      1.004+03
 322.3-03             17.75+00      248.9+00      2.317+03

 251.2+00             794.3+00      2.512+03      7.943+03
-6.476+00            -6.770+00     -70.09+00     -150.0+00
-74.33+00             101.8+00      121.1+00      37.62+00
 2.000+03             1.931+03      1.006+03      1.004+03
 305.9-03             275.6+00      586.1+00      2.985+03

                                                  10.00+03
                                                 -166.0+00
                                                  29.86+00
                                                  1.004+03
                                                  3.811+03
```


Example 1-4.2

Over a frequency range of 8 Hz to 12 Hz using a linear sweep with 0.2 Hz steps, evaluate the transmission function and input impedance of the network shown in Fig. 1-4.3.

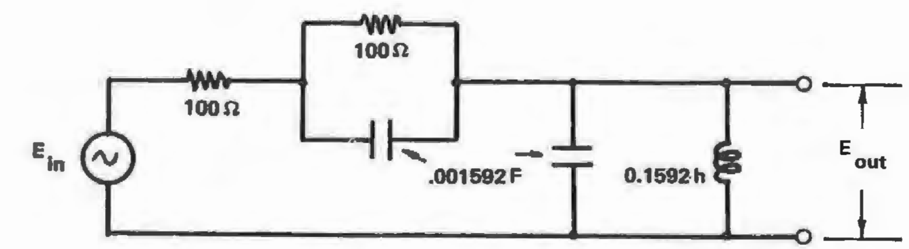


Figure 1-4.3 Network for Example 1-4.2.

The network must be redrawn with the insertion of dummy elements to place it in the ladder format meeting the program input requirements, i.e., only parallel RC or RL networks can be accommodated, not parallel LC networks. The redrawn network is shown in Fig. 1-4.4.

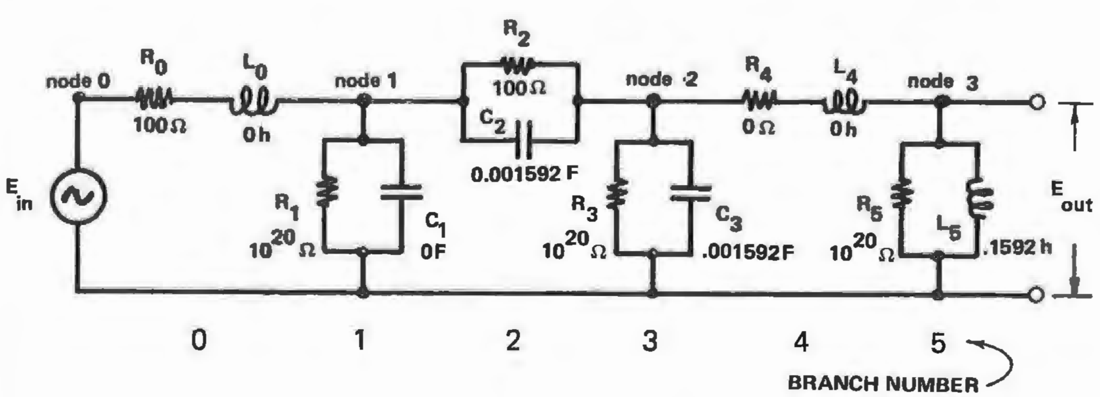


Figure 1-4.4 Network of Fig. 1-4.3 redrawn with dummy elements.

HP-97 PRINTOUT FOR EXAMPLE 1-4.2

PROGRAM INPUT		DATA REVIEW		
3.00 GSBa	enter # of nodes		GSBd	start data review
-1.+20 GSBA	R_5 *	5.000+00	***	branch number
.159155 GSBB	L_5	-100.0+18	***	resistive part *
0. GSBA	R_4	159.2-03	***	reactive part **
0. GSBB	L_4	4.000+00	***	
-1.+20 GSBA	R_3	0.000+00	***	
1.59155-03 GSBC	C_3	0.000+00	***	
-100. GSBA	R_2	3.000+00	***	
1.59155-03 GSBC	C_2	-100.0+18	***	
-1.+20 GSBA	R_1	-1.592-03	***	
0. GSBC	C_1	2.000+00	***	
100. GSBA	R_0	-100.0+00	***	
0. GSBB	L_0	-1.592-03	***	
		1.000+00	***	
		-100.0+18	***	
		0.000+00	***	
		0.000+00	***	
		100.0+00	***	
		0.000+00	***	

* A negative sign indicates parallel connection of elements.

** A negative sign indicates a capacitor.

8. GSBD	start frequency	Analysis Particulars
.2 GSBE	frequency increment	
GSBb	select linear sweep	
GSBe	start analysis	

PROGRAM OUTPUT

8.000+00	freq, Hz	9.000+00	10.00+00	11.00+00
-13.24+00	gain, dB	-7.123+00	-6.158-06	-7.057+00
84.42+00	phase, °	70.22+00	-414.3-06	-58.65+00
101.5+00	Re Z_{in}, Ω	101.2+00	101.0+00	100.8+00
9.915+00	Im Z_{in}, Ω	36.39+00	-13.97+06	-61.40+00
8.200+00		9.200+00	10.20+00	11.20+00
-12.23+00		-5.473+00	-928.2-03	-8.251+00
82.69+00		64.09+00	-21.06+00	-62.31+00
101.5+00		101.2+00	101.0+00	100.8+00
13.01+00		49.15+00	-262.2+00	-52.88+00
8.400+00		9.400+00	10.40+00	11.40+00
-11.13+00		-3.663+00	-2.509+00	-9.306+00
80.60+00		55.[illegible]2+00	-36.38+00	-65.11+00
101.4+00		101.1+00	100.9+00	100.8+00
16.79+00		70.24+00	-137.0+00	-46.76+00
8.600+00		9.600+00	10.60+00	11.60+00
-9.930+00		-1.819+00	-4.173+00	-10.25+00
77.99+00		42.03+00	-46.69+00	-67.31+00
101.3+00		101.1+00	100.9+00	100.7+00
21.55+00		112.1+00	-95.11+00	-42.12+00
8.800+00		9.800+00	10.80+00	11.80+00
-8.602+00		-361.1-03	-5.702+00	-11.09+00
74.66+00		23.05+00	-53.70+00	-69.09+00
101.3+00		101.0+00	100.9+00	100.7+00
27.79+00		237.4+00	-74.08+00	-38.49+00
				12.00+00
				-11.86+00
				-70.55+00
				100.7+00
				-35.55+00

TI-59 PRINTOUT FOR EXAMPLE 1-4.2

DATA REVIEW		PROGRAM OUTPUT	
		0.10	freq, Hz
5.	branch #	-267.18	phase, °
-1. 20	resistive part *	-65.99	gain, dB
0.159155	reactive part **	199.01	Re Z_{in}, Ω
		-9.80	Im Z_{in}, Ω
4.			
0.			
0.		0.13	
		-266.46	
3.		-63.97	
-1. 20		198.44	
-0.00159155		-12.27	
2.			
-100.		0.16	
-0.00159155		-265.57	
		-61.94	
1.		197.55	
-1. 20		-15.30	
0.			
0.		0.20	
100.		-264.47	
0.		-59.89	
		196.17	
		-18.99	
		0.25	
		-263.13	
		-57.82	
		194.06	
		-23.38	
		0.32	
		-261.53	
		-55.70	
		190.91	
		-28.43	

References and Equation Derivation

The algorithm is completely described in the program description section. This particular analysis method is widely referenced. The earliest reference known to the author is T.R. Bashkow [4].

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD RESISTOR VALUE
002	GSB5	odd or even branch?
003	F0?	
004	1/X	if odd numbered branch,
005	F0?	form G = -1/R
006	CHS	
007	STOi	store value
008	RCLI	recall branch # to display
009	RTN	return control to keyboard
010	*LBLC	LOAD CAPACITOR VALUE
011	CHS	change sign of entry
012	*LBLE	LOAD INDUCTOR VALUE
013	GSB5	odd numbered branch?
014	F0?	change sign of entry if
015	CHS	branch number is odd
016	P⇄S	
017	STOi	indirectly store reactive element values
018	P⇄S	
019	DSZI	decrement branch number
020	CF3	clr flag 3 (a NOP statement)
021	RCLI	recall branch number
022	GTO7	goto space and return
023	*LBLD	LOAD START FREQUENCY
024	STO8	
025	GTO7	
026	*LBLE	LOAD FREQUENCY INCREMENT
027	STO9	
028	GTO7	
029	*LBLa	LOAD NUMBER OF NODES
030	STOE	store number of nodes
031	*LBL4	initialization routine
032	EEX	
033	STOA	E_n = 1 + j0
034	CLX	
035	STOB	
036	STOC	
037	STOD	I_{2n} = 0 + j0
038	RCLE	
039	ENT↑	calculate and store
040	+	highest branch number
041	EEX	
042	-	Br# = 2(# nodes) - 1
043	STOI	
044	CF3	clear data entry flag and
045	GTO7	goto space and return
046	*LBLb	SET LINEAR SWEEP
047	CF1	
048	CLX	place "zero" in display
049	GTO7	goto space and return
050	*LBLc	SET LOGARITHMIC SWEEP
051	SF1	
052	EEX	
053	*LBL7	space and return subroutine
054	SPC	
055	RTN	

Step	Key	Comment
056	*LBLd	INPUT DATA REVIEW
057	GSB4	initialize
058	*LBL8	review loop start
059	GSB5	odd numbered branch?
060	RCLI	recall and print branch #
061	PRTX	
062	RCLi	recall branch R or G
063	F0?	
064	1/X	if odd branch (flag 0 set)
065	F0?	form -1/R(i)
066	CHS	
067	PRTX	print branch resistance
068	P⇄S	
069	RCLi	recall branch L or C
070	P⇄S	
071	F0?	change sign if branch odd
072	CHS	
073	PRTX	print L or -C
074	SPC	
075	F3?	test for loop exit
076	RTN	
077	SF3	
078	DSZI	decrement index register and SF3 if index is zero
079	CF3	
080	GTO8	
081	*LBLe	LNAP ANALYSIS START
082	GSB4	goto initialization
083	*LBL9	analysis loop start
084	GSB3	recall shunt branch elements
085	RCLB	recall complex node voltages
086	RCLA	and execute complex multiply
087	GSB1	
088	RCLD	recall previous complex
089	RCLC	branch current and perform
090	GSB2	complex addition
091	STOC	
092	X⇄Y	store complex branch
093	STOD	currents for present branch
094	X⇄Y	
095	CF0	
096	DSZI	decrement index register
097	SF0	& SF0 if index is zero
098	GSB3	recall series branch elts
099	GSB1	execute complex multiply
100	RCLB	recall complex node voltage
101	RCLA	and add to branch voltage
102	GSB2	
103	X⇄Y	store new complex node
104	STOB	voltage
105	X⇄Y	
106	STOA	
107	DSZI	decrement branch number and
108	F0?	test for loop exit
109	GTO9	
110	→P	convert to magnitude & angle

REGISTERS

0 R_0	1 G_1	2 R_2	3 G_3	4 R_4	5 G_5	6 R_6	7 G_7	8 start frequency	9 freq increment
S0 $-C_0$ or L_0	S1 C_1 or $-L_1$	S2 $-C_2$ or L_2	S3 C_3 or $-L_3$	S4 $-C_4$ or L_4	S5 C_5 or $-L_5$	S6 $-C_6$ or L_6	S7 C_7 or $-L_7$	S8 cmplx multiply	S9 cmplx multiply

A Re E_k	B Im E_k	C Re I_k	D Im I_k	E number of nodes	I index

Program Listing II

Step	Key	Comment
111	STOA	temporarily store $\|E_{in}\|$
112	LOG	
113	2	convert to dB
114	0	
115	×	
116	RCL8	
117	RND	recall and print present
118	PRTX	analysis frequency
119	R↓	
120	CHS	recover and print -dB
121	RND	
122	PRTX	
123	R↓	recover phase angle
124	STOB	temporarily store ∡ E_{in}
125	CHS	
126	RND	print -(phase angle)
127	PRTX	
128	RCLD	recall I_0
129	RCLC	
130	→P	
131	RCLA	
132	X⇄Y	
133	÷	
134	X⇄Y	perform complex division:
135	RCLB	
136	X⇄Y	$Z_{in} = E_{in}/I_0$
137	-	
138	X⇄Y	
139	→R	
140	PRTX	print Re Z_{in}
141	X⇄Y	
142	PRTX	print Im Z_{in}
143	PSE	
144	RCL9	recall frequency increment.
145	F1?	
146	ST×8	use multiplicative increment
147	F1?	if logarithmic sweep selected
148	GTOe	
149	ST+8	use additive increment if
150	GTOe	linear sweep selected
151	*LBL1	complex multiplication (a+jb)(c+jd) = e+jf
152	P⇄S	
153	STO8	a
154	STO9	a
155	R↓	
156	ENT↑	
157	R↓	
158	R↓	c
159	ST×8	ac in register 8
160	R↓	d
161	ST×9	ad in register 9
162	×	bd in stack
163	ST-8	ac - bd in register 8
164	R↓	
165	×	bc in stack

Step	Key	Comment
166	ST+9	ad + bc in register 9
167	RCL9	rcl f = ac + bc
168	RCL8	rcl e, ac - bd = e
169	P⇄S	restore register order
170	RTN	return to main program
171	*LBL2	complex add subroutine
172	X⇄Y	
173	R↓	
174	+	
175	R↓	
176	+	
177	R↑	
178	RTN	return to main program
179	*LBL3	complex recall subroutine
180	RCL8	calculate ω = 2πf
181	Pi	
182	×	
183	ENT↑	
184	+	
185	P⇄S	
186	RCLi	recall reactive branch
187	P⇄S	element, b_x, and form
188	×	$2\pi f b_x$
189	X<0?	
190	1/X	form reciprocal if $b_x < 0$
191	RCLi	recall resistive element
192	X<0?	if <0, perform parallel
193	GTO3	series conversion
194	RTN	return to main program
195	*LBL3	parallel series conversion
196	ABS	
197	1/X	conductance⇄resistance
198	X⇄Y	
199	1/X	susceptance⇄reactance
200	X⇄Y	
201	→P	
202	1/X	calculate complex inverse
203	→R	
204	RTN	return to main program
205	*LBL5	odd or even branch subr
206	RCLI	
207	2	form 0 if branch even
208	÷	or 0.5 if branch odd
209	FRC	
210	SF0	
211	X=0?	set flag 0 if branch is odd
212	CF0	
213	R↓	restore x register in stack
214	RTN	return to main program

LABELS					FLAGS
A load R	B load L	C load C	D load start freq	E load freq increment	0 odd branch
a load # of nodes	b set linear sweep	c set log sweep	d start data revu	e start analysis	1 log/lin
0	1 complex multiply	2 complex add	3 complex recall	4 initialize	2
5 odd/even branch	6 series⇄parallel	7 log sweep	8	9	3 data entry

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0		■	DEG	■	FIX	
1		■	GRAD		SCI	
2			RAD		ENG	■
3		■			n 3	

1-4 TI-59 PROGRAM LISTING

000	76	LBL	**LOAD RESISTOR VALUE**
001	11	A	
002	42	STO	temporarily store
003	57	57	
004	71	SBR	set flag 0 if branch number is odd
005	04	04	
006	18	18	
007	43	RCL	recall entry
008	57	57	
009	87	IFF	if odd branch, form G = -1/R
010	00	00	
011	00	00	
012	18	18	
013	72	ST*	store R or G
014	59	59	
015	43	RCL	recall resistor value to display
016	57	57	
017	92	RTN	
018	94	+/-	routine for G = -1/R
019	35	1/X	
020	61	GTO	
021	00	00	
022	13	13	
023	76	LBL	**LOAD CAPACITOR VALUE**
024	13	C	
025	54	)	
026	94	+/-	change sign and temporarily store
027	42	STO	
028	58	58	
029	76	LBL	**LOAD INDUCTOR VALUE**
030	12	B	
031	42	STO	
032	58	58	
033	71	SBR	set flag 0 if branch number is odd
034	04	04	
035	18	18	
036	43	RCL	recall entry
037	58	58	
038	22	INV	if branch number is odd, change the sign of the entry
039	87	IFF	
040	00	00	
041	00	00	
042	44	44	
043	94	+/-	
044	72	ST*	store reactive element
045	56	56	
046	01	1	decrement index of resistive and reactive storage registers
047	94	+/-	
048	44	SUM	
049	59	59	

050	44	SUM	
051	56	56	
052	43	RCL	recall reactive element to display
053	58	58	
054	92	RTN	
055	76	LBL	**LOAD SWEEP STARTING FREQ**
056	14	D	
057	42	STO	
058	55	55	
059	92	RTN	
060	76	LBL	**LOAD FREQ INCREMENT**
061	15	E	
062	42	STO	
063	54	54	
064	92	RTN	
065	76	LBL	**SELECT LINEAR SWEEP**
066	17	B'	
067	22	INV	
068	86	STF	
069	01	01	
070	00	0	display 0
071	92	RTN	
072	76	LBL	**SELECT LOG SWEEP**
073	18	C'	
074	86	STF	
075	01	01	
076	01	1	display 1
077	92	RTN	
078	76	LBL	**INPUT DATA REVIEW**
079	19	D'	
080	71	SBR	initialize
081	03	03	
082	80	80	
083	71	SBR	set flag 0 if branch number is odd
084	04	04	
085	18	18	
086	53	(	recall branch number
087	43	RCL	
088	59	59	
089	75	-	
090	01	1	
091	00	0	
092	54	)	
093	98	ADV	
094	71	SBR	print or display branch number
095	04	04	
096	66	66	
097	73	RC*	recall resistive element
098	59	59	
099	22	INV	

This translation was provided by Mr. Walter Ware

1-4 TI-59 PROGRAM LISTING

100	87	IFF	if odd branch, form R = -1/G
101	00	00	
102	01	01	
103	06	06	
104	35	1/X	
105	94	+/-	
106	71	SBR	display or print resistance
107	04	04	
108	66	66	
109	73	RC*	recall reactive element
110	56	56	
111	22	INV	if odd branch, change sign
112	87	IFF	
113	00	00	
114	01	01	
115	17	17	
116	94	+/-	
117	71	SBR	print or display L or C
118	04	04	
119	66	66	
120	87	IFF	test for loop exit
121	03	03	
122	01	01	
123	47	47	
124	86	STF	decrement indirect storage register indices
125	03	03	
126	01	1	
127	94	+/-	
128	44	SUM	
129	56	56	
130	44	SUM	
131	59	59	
132	43	RCL	set t = 10
133	09	09	
134	85	+	
135	01	1	
136	95	=	
137	32	X:T	
138	43	RCL	recall register index
139	59	59	
140	22	INV	if index = 10, execute one more loop
141	67	EQ	
142	01	01	
143	48	48	
144	61	GTO	
145	00	00	
146	83	83	
147	92	RTN	
148	22	INV	clear flag 3
149	86	STF	

150	03	03	
151	68	NOP	
152	68	NOP	
153	68	NOP	
154	61	GTO	
155	00	00	goto loop start
156	83	83	
157	29	CP	complex recall subr
158	53	(	
159	43	RCL	calculate $\omega=2\pi f$
160	55	55	
161	65	×	
162	89	π	
163	65	×	
164	02	2	
165	65	×	recall reactive branch element value and form branch immittance
166	73	RC*	
167	56	56	
168	54	)	
169	77	GE	if immittance is negative, form reciprocal
170	01	01	
171	73	73	
172	35	1/X	
173	42	STO	store immittance
174	58	58	
175	73	RC*	recall branch resistance and store
176	59	59	
177	42	STO	
178	57	57	
179	22	INV	if resistance negative, perform series⇄parallel conversion
180	77	GE	
181	01	01	
182	84	84	
183	92	RTN	
184	43	RCL	series⇄parallel conversion subroutine
185	57	57	
186	50	I×I	conductance⇄resistance
187	35	1/X	
188	32	X:T	
189	43	RCL	susceptance⇄reactance
190	58	58	
191	35	1/X	
192	94	+/-	
193	22	INV	calculate complex inverse
194	37	P/R	
195	94	+/-	
196	32	X:T	
197	35	1/X	
198	32	X:T	
199	37	P/R	

1-4 TI-59 PROGRAM LISTING

200	42	STO	temporarily store immittance
201	58	58	
202	32	X:T	temporarily store resistance or cond
203	42	STO	
204	57	57	
205	92	RTN	return to main program
206	76	LBL	LNAP ANALYSIS START
207	10	E'	
208	58	FIX	set display mode
209	02	02	
210	98	ADV	advance paper
211	98	ADV	
212	71	SBR	initialize
213	03	03	
214	83	83	
215	71	SBR	recall shunt branch
216	01	01	
217	57	57	
218	43	RCL	recall complex node voltage and execute complex multiply to obtain complex branch current
219	57	57	
220	42	STO	
221	01	01	
222	43	RCL	
223	58	58	
224	42	STO	
225	02	02	
226	43	RCL	
227	49	49	
228	42	STO	
229	03	03	
230	43	RCL	
231	50	50	
232	42	STO	
233	04	04	
234	36	PGM	
235	04	04	
236	13	C	
237	43	RCL	recall previous complex branch current, perform complex add and store result
238	01	01	
239	44	SUM	
240	51	51	
241	43	RCL	
242	02	02	
243	44	SUM	
244	52	52	
245	01	1	decrement indirect recall indices
246	94	+/-	
247	44	SUM	
248	56	56	
249	44	SUM	

250	59	59	
251	71	SBR	recall series branch elements
252	01	01	
253	57	57	
254	43	RCL	multiply series impedance by complex branch current to obtain series branch voltage drop
255	57	57	
256	42	STO	
257	01	01	
258	43	RCL	
259	58	58	
260	42	STO	
261	02	02	
262	43	RCL	
263	51	51	
264	42	STO	
265	03	03	
266	43	RCL	
267	52	52	
268	42	STO	
269	04	04	
270	36	PGM	
271	04	04	
272	13	C	
273	43	RCL	add complex series voltage drop to previous node voltage to obtain next node voltage and store result
274	01	01	
275	44	SUM	
276	49	49	
277	43	RCL	
278	02	02	
279	44	SUM	
280	50	50	
281	01	1	decrement indirect recall indices
282	94	+/-	
283	44	SUM	
284	56	56	
285	44	SUM	
286	59	59	
287	43	RCL	test for loop exit
288	09	09	
289	32	X:T	
290	43	RCL	
291	59	59	
292	67	EQ	
293	02	02	
294	98	98	
295	61	GTO	repeat loop
296	02	02	
297	15	15	
298	43	RCL	recall present freq
299	55	55	

1-4 TI-59 PROGRAM LISTING

300	71	SBR	print or display frequency
301	04	04	
302	66	66	
303	43	RCL	recall complex input voltage
304	49	49	
305	32	X:T	
306	43	RCL	
307	50	50	
308	22	INV	convert to polar
309	37	P/R	
310	94	+/-	change sign of angle and store
311	42	STO	
312	05	05	
313	71	SBR	print or display angle of network transmission function
314	04	04	
315	66	66	
316	32	X:T	calculate 20 log of network transmission function magnitude
317	28	LOG	
318	94	+/-	
319	65	×	
320	02	2	
321	00	0	
322	95	=	
323	42	STO	
324	06	06	
325	71	SBR	print or display dB response
326	04	04	
327	66	66	
328	43	RCL	recall complex network input voltage
329	49	49	
330	42	STO	
331	01	01	
332	43	RCL	
333	50	50	
334	42	STO	
335	02	02	
336	43	RCL	recall complex network input current
337	51	51	
338	42	STO	
339	03	03	
340	43	RCL	
341	52	52	
342	42	STO	
343	04	04	
344	36	PGM	perform complex division
345	04	04	
346	18	C'	
347	43	RCL	print or display Re Z_{in}
348	01	01	
349	71	SBR	

350	04	04	
351	66	66	
352	43	RCL	print or display Im Z_{in}
353	02	02	
354	71	SBR	
355	04	04	
356	66	66	
357	43	RCL	recall frequency increment
358	54	54	
359	87	IFF	jump if log sweep selected
360	01	01	
361	03	03	
362	68	68	
363	44	SUM	add frequency increment for linear sweep
364	55	55	
365	61	GTO	
366	02	02	
367	06	06	
368	49	PRD	multiply by frequency increment for log sweep
369	55	55	
370	61	GTO	
371	02	02	
372	06	06	
373	76	LBL	LOAD NUMBER OF NODES
374	16	A'	
375	71	SBR	test for printer
376	04	04	
377	75	75	
378	42	STO	store number of nodes
379	53	53	
380	09	9	initialization subr
381	42	STO	set minimum loop counter value allowed
382	09	09	
383	53	(	calculate highest branch number storage index for real immittance storage
384	43	RCL	
385	53	53	
386	65	×	
387	02	2	
388	75	-	
389	01	1	
390	85	+	
391	01	1	
392	00	0	
393	54	)	
394	42	STO	
395	59	59	
396	85	+	calculate highest branch number storage index for imaginary immitance storage
397	01	1	
398	00	0	
399	54	)	

1-4 TI-59 PROGRAM LISTING

400	42	STO	
401	56	56	
402	01	1	initialize node voltage of highest node: $E_n = 1 + j0$
403	42	STO	
404	49	49	
405	00	0	
406	42	STO	
407	50	50	
408	42	STO	initialize $I_{2n} = 0 + j0$
409	51	51	
410	42	STO	
411	52	52	
412	22	INV	clear flag 3
413	86	STF	
414	03	03	
415	43	RCL	recall number of nodes
416	53	53	
417	92	RTN	return to main program
418	29	CP	odd or even branch subr
419	22	INV	clear flag 0
420	86	STF	
421	00	00	
422	43	RCL	
423	59	59	
424	55	÷	
425	02	2	
426	54	)	
427	22	INV	
428	59	INT	
429	67	EQ	
430	04	04	
431	34	34	
432	86	STF	set flag 0 if branch number is odd
433	00	00	
434	92	RTN	
435	00	0	unused program memory
436	00	0	
437	00	0	
438	00	0	
439	00	0	
440	00	0	
441	00	0	
442	00	0	
443	00	0	
444	00	0	
445	00	0	
446	00	0	
447	00	0	
448	00	0	
449	00	0	
450	00	0	
451	00	0	
452	00	0	
453	00	0	
454	00	0	
455	00	0	
456	00	0	
457	00	0	
458	00	0	
459	00	0	
460	00	0	
461	00	0	
462	00	0	
463	00	0	
464	00	0	
465	00	0	
466	22	INV	print or R/S routine
467	87	IFF	
468	05	05	
469	04	04	
470	73	73	
471	91	R/S	
472	92	RTN	
473	99	PRT	
474	92	RTN	
475	69	OP	printer sense routine
476	08	08	
477	86	STF	
478	05	05	
479	92	RTN	

REGISTER ALLOCATIONS FOR TI-59

register number	contents
0	
1	Re — complx arith temp storage
2	Im — complx arith temp storage
3	Re — complx arith temp storage
4	Im — complx arith temp storage
5	
6	xmsn fcn magnitude
7	
8	
9	loop counter
10–19	real immittance storage
20–29	imaginary immittance storage
30	
31	
32	
33	
34	
35	
36	
37	
38	
39	
40	
40	
41	
42	
43	
44	
45	
46	
47	
48	
49	Re node V sum
50	Im node V sum
51	Re branch I
52	Im branch I
53	# of nodes
54	freq increment
55	start freq
56	Im storage index
57	temp store
58	temp store
59	Re storage index

PROGRAM 1-5 LC - L N A P , LC LADDER NETWORK ANALYSIS PROGRAM.

Program Description and Equations Used

This program evaluates the frequency response and input impedance of a resistively terminated lossless (LC) ladder network having up to seven branches. The frequency response is provided as magnitude (dB) and phase (degrees, radians, or grads), and the input impedance is provided as real and imaginary parts.

The input impedance is the impedance seen by the voltage generator in the source. It is more common to calculate the input impedance at the input terminals of the lossless ladder network, but this way was not implemented because program steps are not available for the coding to recall the source resistor value and subtract it from the real part of the input impedance. If the program feature of allowing the number of branches to be entered via a user definable key (key "a") is sacrificed, and the number of branches is stored into register E, then the additional coding for calculating the network input impedance can be added to the program by deleting steps 028 and 029 and adding "RCLØ," "-" after step Ø97 (Ø99 before deletions).

The frequency response and input impedance evaluation frequency can be incremented in either a linear manner using an additive increment, or a logarithmic manner using a multiplicative increment. Each branch of the network may contain an inductor (L), a capacitor (C), a series LC network, or a parallel LC network. All element values and interconnection topology are stored, and can be reviewed at any time to check or correct the component values or interconnection.

Because of the available number of HP-67/97 registers, the number of branches cannot exceed seven. The TI-59 can accommodate data for 20 branches. Elements that inhibit signal flow through the network are not allowed, and will cause the program execution to halt displaying "Error." Examples of elements that inhibit signal flow are single shunt resistors or inductors that have zero value, or series capacitors in series branches that have zero value.

The algorithm used by this program is the same as used in Program 1-4 where 1 volt is assumed at the network output, and the required input voltage is calculated. In this program, the branch immittances (impedances or admittances) are purely imaginary, and the branch numbers start with branch #1 instead of branch #0. This changes all indices by +1. The difference is necessary to let the DSZ instruction operation allow the source resistance to be added to branch #1 with minimum coding. The load resistance is combined with the last branch immittance. If the number of branches is odd, the last branch consists of the load resistor alone.

1-5

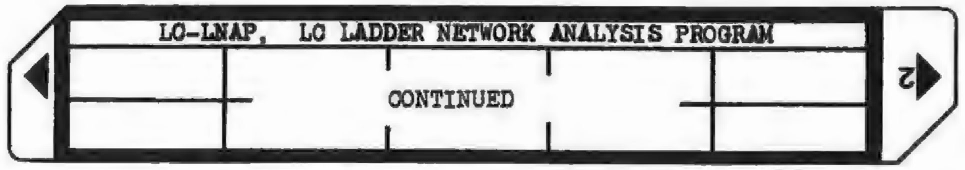
User Instructions

LC-LNAP, LC LADDER NETWORK ANALYSIS PROGRAM				
load # of branches	linear sweep?	log sweep?	review input data	start analysis
load $R_L \uparrow R_S$	load br C	load br L	load start frequency	load freq increment

2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Load the number of branches in the network	# branches	f A	
3	Enter the load and source resistances	R_L	ENT↑	
	in ohms	R_S	A	
4	Load branch capacitance:			
	If a parallel tank in a series branch,			
	or a series tank in a shunt branch,			
	change the sign of the mantissa in			
	the capacitor value	$\pm C_{branch}$	B	
		(farads)		
	Start loading network capacitors			
	(and inductors) from the highest			
	numbered branch (load resistor end)			
5	Load branch inductance:			
	If a parallel tank in a series branch,			
	or a series tank in a shunt branch,			
	change the sign of the mantissa in			
	the inductor value	$\pm L_{branch}$	C	
		(henries)		
6	Input data review (optional)		f D	R_{load}
				space
	Negative element values indicate series			highest branch #
	tanks in shunt branches, or parallel			$\pm C$
	tanks in series branches			$\pm L$
				space
				:
				R_{source}
7	Select frequency sweep mode:			
	a) linear sweep		f B	
	b) logarithmic sweep		f C	
8	Load start frequency for sweep in hertz	f_{start}	D	

1-5 User Instructions



STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
9	Load frequency increment	f_{incr}	[E]	
	If linear sweep, the increment is added to the present frequency to obtain the next frequency.			
	If logarithmic sweep, the increment is multiplied by the present frequency to obtain the next frequency.			
10	Start analysis		[f] [E]	freq (Hz)
				gain (dB)
	* The phase units will be in whatever trig mode the calculator is set. The trig mode is at the discretion of the user.			phase*( )
				Re Z_{in}, Ω
				Im Z_{in}, Ω
				space
				⋮
11	Stop analysis: Press R/S when the printer starts to print.			
	Pressing R/S at other times may leave the registers interchanged. To determine if an interchange has occurred, goto step 6 and review input data. If L and C values are reversed, execute a P⇄S instruction from the keyboard.			

Example 1-5.1

Bartlett's bisection theorem [53], [56], [57] has been applied to an equally terminated (1000 ohm) third order Butterworth bandpass filter with 10 kHz center frequency and 1 kHz bandwidth to produce the unequally terminated LC filter shown in Fig. 1-5.1. The source resistance is 1000 ohms and the load resistance is 10000 ohms. Determine the frequency response and input impedance of this LC network over a frequency range of 9000 Hz to 10900 Hz using a linear sweep with 100 Hz steps.

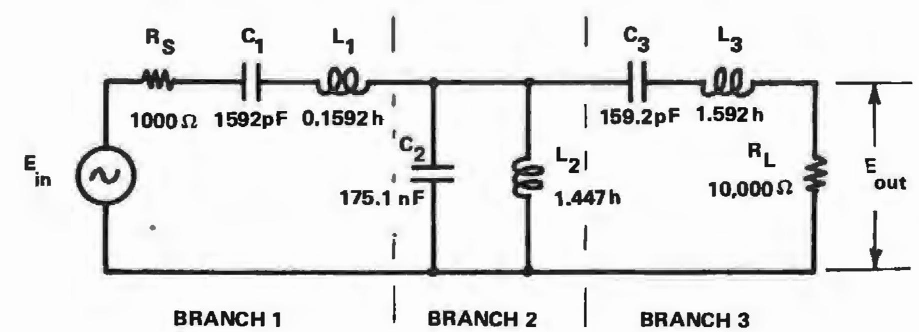


Figure 1-5.2 Network for Example 1-5.1.

PROGRAM INPUT			DATA REVIEW		
3.00	GSBa	number of branches		GSBd	start review
			10.00+03	***	load resistance
10000.	ENT↑	R_L			
1000.	GSBA	R_S	3.000+00	***	branch number
			159.2-12	***	C
159.2-12	GSBB	C_3	1.592+00	***	L
1.592	GSBC	L_3			
			2.000+00	***	branch number
.1751-06	GSBB	C_2	175.1-09	***	C
1.447-03	GSBC	L_2	1.447-03	***	L
1592.-12	GSBB	C_1	1.000+00	***	branch number
.1592	GSBC	L_1	1.592-09	***	C
			159.2-03	***	L
	GSBb	linear sweep	1.000+03	***	source resistance
9000.00	GSBD	start freq , Hz			
100.00	GSBE	freq incr , Hz			

HP-97 PRINTOUT FOR EXAMPLE 1-5.1

GSBe **start analysis**

PROGRAM OUTPUT

9000.00	freq, Hz	9500.00	10000.00	10500.00
-20.29	gain, dB	-4.14	-0.83	-3.58
-146.92	phase, °	138.09	-0.48	-132.20
1003.54	Re Z_{in}, Ω	1042.21	10994.89	1048.16
-1666.97	Im Z_{in}, Ω	-93.28	-227.42	10.31
9100.00		9600.00	10100.00	10600.00
-17.44		-1.93	-0.83	-6.35
-154.43		106.39	-23.45	-157.06
1005.32		1083.75	2916.29	1027.54
÷1391.66		364.31	-3831.85	363.73
9200.00		9700.00	10200.00	10700.00
-14.32		-1.04	-0.84	-9.45
-164.23		74.86	-47.28	-175.59
1008.28		1186.21	1504.60	1016.80
-1105.20		979.39	-1965.91	668.16
9300.00		9800.00	10300.00	10800.00
-10.93		-0.85	-1.01	-12.47
-177.51		47.27	-73.45	170.95
1013.48		1498.05	1195.73	1010.80
-801.44		1951.68	-1020.61	941.35
9400.00		9900.00	10400.00	10900.00
-7.40		-0.83	-1.76	-15.27
163.89		22.71	-102.68	160.95
1023.10		2964.18	1091.62	1007.25
-470.17		3870.77	-426.27	1193.23

Example 1-5.2

The filter shown in Fig. 1-5.3 is a 5th order, 30° modular angle, 50% reflection coefficient elliptic filter designed for 10 kHz cutoff frequency and 1000 ohm impedance level. This example shows how dummy elements are inserted to place the filter in proper ladder format for this program. The frequency response and input impedance are calculated with the analysis frequency being logarithmically swept from 1 kHz to 100 kHz using 10 points per decade.

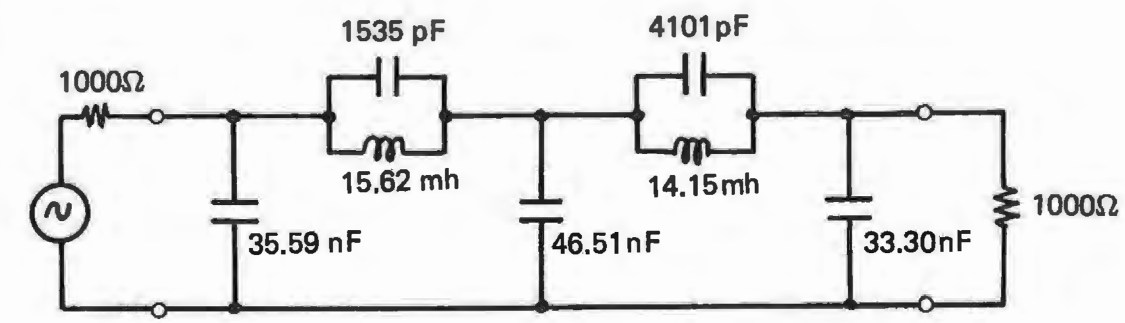


Figure 1-5.3 Elliptic filter for Example 1-5.2.

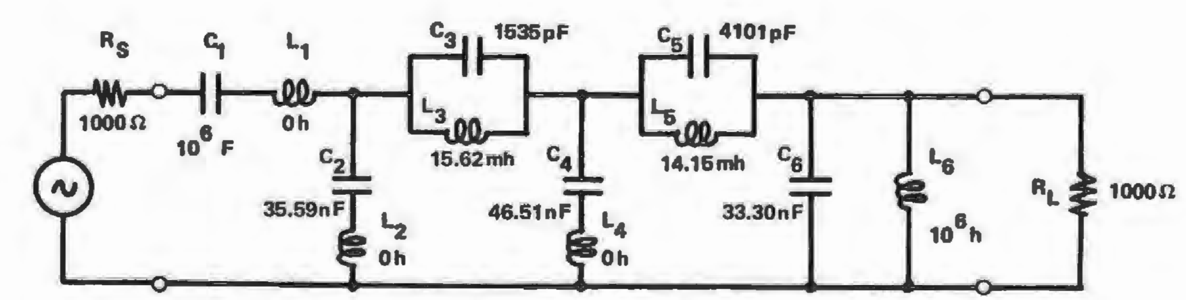


Figure 1-5.4 Network of Fig. 1-5.3 redrawn with dummy elements to place in proper ladder format for program input.

HP-97 PRINTOUT FOR EXAMPLE 1-5.2

PROGRAM INPUT

6.00	GSBa	# of branches
1000.	ENT↑	enter load R
	GSBA	enter source R
.03330-06	GSBB	C_6
1.+06	GSBC	L_6 (dummy)
-4101.-12	GSBB	C_5 (note minus)
-14.15-03	GSBC	L_5 " "
-.04651-06	GSBB	C_4
0.	GSBC	L_4 (dummy)
-1535.-12	GSBB	C_3 (note minus)
-15.62-03	GSBC	L_3 " "
-.03559-06	GSBB	C_2
0.	GSBC	L_2 (dummy)
1.+06	GSBB	C_1 (dummy)
0.	GSBC	L_1 (dummy)
	GSBc	log sweep
1000.	GSBD	start freq
.10	10^x	increment for
	GSBE	10 points per decade

DATA REVIEW

	GSBd	start review
1.000+03	***	load resistance
6.000+00	***	branch number
33.30-09	***	C
1.000+06	***	L
5.000+00	***	branch number
-4.101-09	***	C
-14.15-03	***	L
4.000+00	***	branch number
-46.51-09	***	C
0.000+00	***	L
3.000+00	***	branch number
-1.535-09	***	C
-15.62-03	***	L
2.000+00	***	branch number
-35.59-09	***	C
0.000+00	***	L
1.000+00	***	branch number
1.000+06	***	C
0.000+00	***	L
1.000+03	***	source resistance

HP-97 PRINTOUT FOR EXAMPLE 1-5.2

DSP3	set display format
GSBe	start analysis

PROGRAM OUTPUT

1000.000	freq, Hz	3162.278	10000.000	31622.777
-6.304	gain, dB	-7.265	-7.275	-95.872
-25.492	phase, °	-71.409	46.462	106.137
1731.945	Re Z_{in}, Ω	1341.972	2425.429	1000.000
-354.815	Im Z_{in}, Ω	-144.944	-1314.534	-141.760
1258.925		3981.072	12589.254	39810.717
-6.445		-7.140	-29.301	-80.704
-31.679		-87.812	-34.999	-77.360
1641.443		1354.476	1001.496	1000.000
-367.449		-16.643	-520.507	-110.753
1584.893		5011.872	15848.932	50118.724
-6.636		-6.543	-48.256	-77.037
-39.149		-111.901	-53.026	-80.045
1545.329		1510.069	1000.017	1000.000
-354.383		144.638	-334.325	-87.088
1995.262		6309.573	19952.623	63095.735
-6.872		-6.035	-75.897	-76.495
-48.061		-151.933	-62.745	-82.135
1455.184		2050.032	1000.000	1000.000
-312.318		-105.785	-242.424	-68.745
2511.886		7943.282	25118.864	79432.824
-7.114		-7.176	-76.839	-77.128
-58.642		154.808	110.795	-83.773
1383.210		1466.092	1000.000	1000.000
-242.073		-532.050	-183.485	-54.393
				100000.000
				-78.338
				-85.064
				1000.000
				-43.101

Program Listing I

Step	Key	Comments
001	*LBLA	LOAD $R_L \uparrow R_s$
002	STO0	store R_s
003	R↓	
004	P⇄S	
005	STO0	store R_L
006	P⇄S	
007	GTO7	goto space and return
008	*LBLC	LOAD BRANCH INDUCTANCE
009	CHS	indicate inductance by chs
010	P⇄S	interchange registers and
011	GSB6	goto capacitor load routine
012	P⇄S	
013	DSZI	decrement and recall branch
014	RCLI	
015	GTO7	goto space and return
016	*LBLB	LOAD BRANCH CAPACITANCE
017	GSB5	odd/even branch?
018	F0?	change sign of entry if
019	CHS	branch number is odd
020	STOi	store entry
021	RTN	return control to keyboard
022	*LBLD	LOAD START FREQUENCY
023	STO8	
024	GTO7	
025	*LBLE	LOAD FREQUENCY INCREMENT
026	STO9	
027	GTO7	
028	*LBLa	LOAD NUMBER OF BRANCHES
029	STOE	
030	*LBL4	initialization routine
031	EEX	
032	STOA	$E_n = 1 + j0$
033	CLX	
034	STOB	
035	STOC	
036	STOD	$I_{2n+1} = 0 + j0$
037	RCLE	set index to
038	STOI	highest branch number
039	SF2	initialize flags
040	CF3	
041	GTO7	goto space and return
042	*LBLb	SELECT LINEAR SWEEP
043	CF1	
044	GTO7	
045	*LBLc	SELECT LOGARITHMIC SWEEP
046	SF1	
047	GTO7	
048	*LBLe	START ANALYSIS
049	GSB4	initialize
050	*LBL9	analysis loop start
051	GSB3	recall shunt branch Y
052	RCLB	recall complex node voltage
053	RCLA	
054	GSB1	calculate shunt branch I
055	RCLC	recall next higher (series)
056	RCLD	branch current
057	GSB2	add complex branch currents
058	STOC	store next lower branch
059	X⇄Y	current (complex)
060	STOD	
061	DSZI	decrement branch number
062	GSB3	recall series branch Z
063	CLX	
064	RCL0	
065	CF0	
066	DSZI	If branch 1, add
067	SF0	source resistance to
068	F0?	branch impedance
069	CLX	
070	ENT↑	
071	RCLD	recall present
072	RCLC	branch current
073	GSB1	calculate branch voltage
074	RCLA	recall next higher
075	RCLB	branch voltage
076	GSB2	add branch voltages
077	STOA	
078	X⇄Y	store next lower
079	STOB	node voltage
080	F0?	test for loop exit
081	GTO9	
082	X⇄Y	
083	→P	convert to magnitude & angle
084	LOG	
085	2	calculate magnitude in dB
086	0	
087	×	
088	RCL8	recall present frequency
089	SF0	indicate sign change in p/o
090	GSB0	gosub printout (p/o) routine
091	RCLD	
092	CHS	
093	RCLC	recall branch 1 current (I_0)
094	→P	and form complex inverse
095	1/X	
096	→R	
097	RCLB	recall node 0 voltage (E_{in})
098	RCLA	
099	GSB1	perform complex multiply
100	PRTX	print Re Z_{in}
101	X⇄Y	
102	PRTX	print Im Z_{in}
103	RCL9	recall frequency increment
104	F1?	
105	ST×8	multiply present frequency
106	F1?	by increment if log sweep
107	GTOe	
108	ST+8	add increment to present
109	GTOe	frequency if linear sweep
110	*LBLd	INPUT DATA REVIEW
111	GSB4	initialize registers & flags
112	P⇄S	

REGISTERS

0 R_s	1 C_1	2 C_2	3 C_3	4 C_4	5 C_5	6 C_6	7 C_7	8 present frequency	9 freq increment
S0 R_L	S1 L_1	S2 L_2	S3 L_3	S4 L_4	S5 L_5	S6 L_6	S7 L_7	S8 cmplx multiply	S9 cmplx multiply

A Re E_k	B Im E_k	C Re I_j	D Im I_j	E number of branches	I index

Program Listing II

Step	Key	Comments
113	RCL0	
114	P⇄S	
115	PRTX	
116	*LBL8	data review loop start
117	GSB5	odd/even branch?
118	P⇄S	
119	RCLi	recall branch inductance
120	P⇄S	
121	CHS	
122	RCLi	recall branch capacitance
123	RCLI	recall branch number
124	SPC	
125	GSB0	gosub printout routine
126	DSZI	decrement branch number
127	GTO8	and exit at branch 0
128	RCL0	recall and print
129	SPC	source resistance
130	PRTX	
131	*LBL7	space and return subroutine
132	SPC	
133	RTN	
134	*LBL0	output subroutine
135	PRTX	print x register contents
136	GSB0	print y register contents
137	*LBL0	print z register contents
138	R↓	
139	F0?	if odd branch, change sign
140	CHS	
141	PRTX	
142	RTN	return to subroutine call
143	*LBL1	complex multiplication
144	P⇄S	(a+jb)(c+jd) = e+jf
145	STO8	a
146	STO9	a
147	R↓	b
148	ENT↑	b
149	R↓	b
150	R↓	c
151	ST×8	ac
152	R↓	d
153	ST×9	ad
154	×	bd
155	ST-8	ac - bd = e
156	R↓	b
157	×	bc
158	ST+9	ad + bc = f
159	RCL9	recall f
160	RCL8	recall e
161	P⇄S	
162	RTN	return to subroutine call
163	*LBL2	complex add: (b↑a↑c↑d)
164	R↓	(a+jb) + (c+jd) = e+jf
165	+	a + c = e
166	R↓	
167	+	b + d = f
168	R↑	
169	RTN	
170	*LBL3	branch immittance recall
171	GSB5	odd/even branch?
172	RCL8	
173	Pi	
174	×	form $\omega = 2\pi f_{present}$
175	ENT↑	
176	+	
177	ENT↑	
178	ENT↑	
179	RCLi	recall C_i
180	X<0?	set flag 3 if C_i negative
181	SF3	
182	×	form ωC_i
183	X<0?	if C_i minus, take reciprocal
184	1/X	
185	X⇄Y	
186	P⇄S	
187	RCLi	recall L_i
188	P⇄S	
189	×	form ωL_i
190	X<0?	if L_i minus, take reciprocal
191	1/X	
192	+	form branch immittance
193	F0?	if odd branch (exclusive or)
194	F3?	sign of C_i negative, form
195	F3?	-1/(immittance)
196	GSB3	
197	F2?	add load resistance if
198	GTO0	first time through loop
199	0	
200	RTN	return to subroutine call
201	*LBL3	negative reciprocal routine
202	1/X	
203	CHS	
204	RTN	
205	*LBL0	load resistance addition
206	F0?	subroutine
207	0	if odd branch, reactive part
208	P⇄S	does not exist.
209	RCL0	recall load resistance and
210	P⇄S	form load conductance
211	1/X	
212	F0?	if odd branch, increment
213	ISZI	index register
214	RTN	return to subroutine call
215	*LBL5	odd/even branch subroutine
216	RCLI	
217	2	form 0 if branch even or
218	÷	0.5 if branch odd
219	FRC	
220	SF0	
221	X=0?	set flag 0 if branch odd
222	CF0	
223	R↓	restore stack x register
224	RTN	return to subroutine call

LABELS

A load $R_L \uparrow R_s$	B load C_i	C load L_i	D load start freq	E load freq incr
a # of branches	b linear sweep	c log sweep	d data review	e start analysis
0 multiple uses	1 complex multiply	2 complex add	3 complex recall	4 initialize
5 odd/even branch	6	7 space & return	8 input data loop	9 analysis loop

FLAGS

0 odd br
1 log swp
2 first time thru loop
3

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
			users choice	
0		■	DEG	FIX
1		■	GRAD	SCI
2		■	RAD	ENG
3				n____

PROGRAM 1-6 EQUIVALENT INPUT NOISE OF AN AMPLIFIER WITH GENERALIZED INPUT COUPLING NETWORK

Program Description and Equations Used

When low noise amplifiers are designed, the amplifier equivalent current and voltage noise densities (noise in a 1 Hz band), and the coupling network noise sources, response, and impedance behavior must be considered. This program calculates the total noise voltage density that is reflected to the amplifier input which is coupled to a sensor by means of a transformer (Fig. 1-6.1).

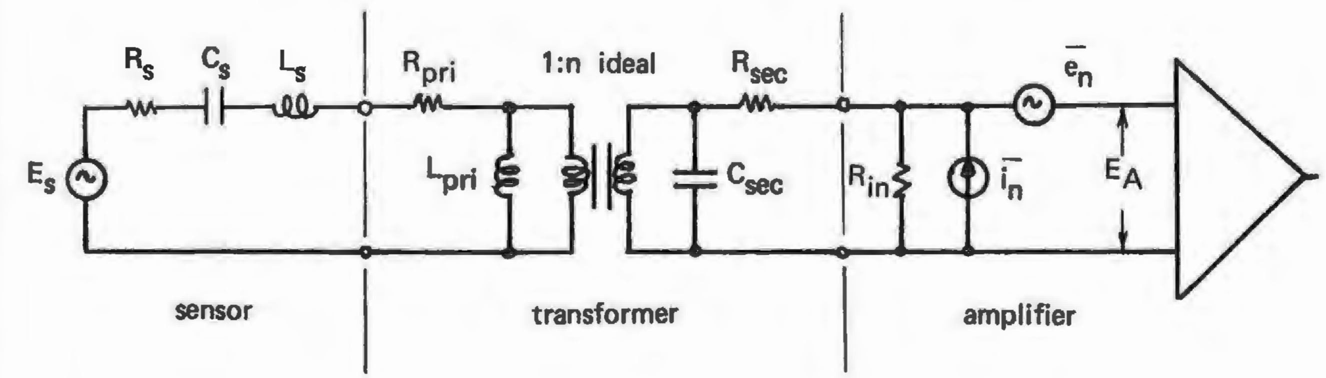


Figure 1-6.1 Generalized input coupling network.

The transformer model includes the turns ratio (1:n), the primary and secondary resistances (R_{pri} and R_{sec}), the primary inductance (L_{pri}), and the secondary capacitance (C_{sec}). The coupling network noise sources include: the thermal noise densities (Johnson noise) of the transformer primary and secondary resistances and of the source resistance, the amplifier equivalent voltage noise density ($\bar{e}_n$), and the equivalent noise voltage density generated by the amplifier current noise density ($\bar{i}_n$) flowing through the coupling network impedance presented to the amplifier input.

The noise voltage density of each noise source is reflected to the amplifier input through the network gain (at the analysis frequency) from the noise source location to the amplifier input. The total noise reflected to the amplifier input is calculated from the root-sum-squared (RSS) values of the individual contributions.

The sensor is represented by a voltage source (E_s) and a series LRC network (L_s, R_s, and C_s). The inductance may be set to zero if not needed, and the capacitor may be set to 10^{50} farads to remove its contribution. The sensor resistance may be zero if the transformer primary resistance is not zero and vice-versa.

The equivalent circuit can be modified to reflect the transformer secondary capacitance to the primary if desired by deleting steps 059, 060, and 061 in the program. The primary capacitance is now loaded in step 2f of the users' instructions. This modification allows piezoelectric transducer elements to be modeled as the source. R_{pri} is set to zero, and the transformer primary capacitance is used to represent the clamped capacity of the piezoelectric element.

If the transformer is not wanted in the circuit, the turns ratio should be set to one.

The equations are derived using nodal analysis, and the user is referred to the section following Example 1-6.2 for details.

User Instructions

AMPLIFIER EQUIVALENT INPUT NOISE					
print ? / select dB output	select linear output	enter noise current density	enter noise voltage density	enter frequency & compute	2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of the program card			
2	Load network element values			
	a) sensor resistance, ohms	R_s	STO 0	
	b) sensor capacitance, farads	C_s	STO 1	
	c) sensor inductance, henries	L_s	STO 2	
	d) transformer primary resistance, Ω	R_{pri}	STO 3	
	e) transformer primary inductance, h	L_{pri}	STO 4	
	f) xfmr secondary capacitance, farads	C_{sec}	STO 5	
	g) xfmr secondary resistance, ohms	R_{sec}	STO 6	
	h) amplifier input resistance, ohms	R_{in}	STO 7	
	i) transformer turns ratio	n	STO 8	
3	Select output mode			
	a) for voltages in dBV and network gain in dB		A	
	b) for voltages in volts, and network gain as a voltage ratio		B	
4	Select print(1) / run-stop (0) option		f A	0,1
5	Enter amplifier input noise current density	$\overline{i_n}$, A/$\sqrt{Hz}$	C	
6	Enter amplifier input noise voltage density	$\overline{e_n}$, V/$\sqrt{Hz}$	D	
7	Enter analysis frequency and compute output	f, Hz	E	gain
				space
	Note:			$\overline{e_1}$
	All noise voltages are reflected			$\overline{e_2}$
	to the amplifier input, i.e., the			$\overline{e_3}$
	gain of the network from the noise			$\overline{e_n}$, amp
	voltage source to the amplifier			$\overline{i_n}$*Z, amp
	input is taken into account.			space
				RSS noise
				space
				space
8	For another case, go back to steps 2 thru 6 as required			

Example 1-6.1

A type 2N4867A low-noise field effect transistor (FET) is to be used as a preamplifier for a piezoelectric hydrophone. A frequency range of 10 Hz to 1000 Hz is to be covered. The hydrophone is operating well below its self-resonant frequency, hence, its equivalent circuit is accurately represented by a 4000 pF capacitor in series with a 10 ohm resistor. To avoid preamplifier overload problems from cable flutter and other subsonic signals, the input resistance of the preamplifier is chosen to provide a 50 Hz low frequency break with the hydrophone capacity. The hydrophone will be coupled to the preamp without using a transformer, therefore a dummy turns ratio of 1:1 will be used in the program. The current and voltage noise densities for the 2N4867A are listed in Table 1-6.1.

Table 1-6.1 Current and voltage noise densities of 2N4867A operating at drain current I_{dss}.

Frequency, Hz	$\bar{i}_n$, noise $A/\sqrt{Hz}$	$\bar{e}_n$, noise $V/\sqrt{Hz}$
10	6×10^{-16}	7.0×10^{-9}
20	6×10^{-16}	5.3×10^{-9}
50	6×10^{-16}	4.1×10^{-9}
100	6×10^{-16}	3.6×10^{-9}
200	6.1×10^{-16}	3.2×10^{-9}
500	6.2×10^{-16}	2.8×10^{-9}
1000	6.3×10^{-16}	2.7×10^{-9}

The HP-97 printout is shown on the next page. Dummy values have been entered for unused components to remove their contribution.

HP-97 PRINTOUT FOR EXAMPLE 1-6.1

```
PROGRAM INPUT
    10.0 ST00  sensor resistance
   4.-09 ST01  sensor capacitance
     0.0 ST02  sensor inductance
     0.0 ST03  primary resistance
   1.+50 ST04  primary inductance
     0.0 ST05  secondary capacitance
     1.0 ST06  secondary resistance
         RCL1  }
    50.0    x  }
           Pi  }  calculate and store
            x  }  amplifier input
     2.0    x  }  resistance for
          1/X  }  50 Hz  breakpoint
795774.7  ***  }
         ST07  }
     1.0 ST08  n, xfmr turns ratio

         GSBA  select dB/dBV output

PROGRAM OUTPUT
   6.-16 GSBC  i_n @ 10 Hz
   7.-09 GSBD  e_n @ 10 Hz
    10.0 GSBE  frequency

   -14.1  ***  A_v, network gain, dB

  -202.1  ***  R_s+R_pri thermal noise, dBV
  -212.1  ***  R_sec thermal noise, dBV
  -139.1  ***  R_in thermal noise, dBV
  -163.1  ***  e_n, transistor
  -186.6  ***  i_n*Z equiv noise, dBV

  -139.1  ***  total noise (RSS), dBV

 5.3-09 GSBD  e_n @ 20 Hz  *
   20.0 GSBE  frequency

   -6.6  ***
                     * i_n is unchanged
 -196.5  ***           from the previous
 -206.5  ***           entry.
 -139.5  ***
 -165.5  ***
 -187.1  ***

 -139.5  ***  total noise at 20 Hz

 4.1-09 GSBD  e_n @ 50 Hz
   50.0 GSBE  frequency

   -3.0  ***

 -198.9  ***
 -200.9  ***
 -141.9  ***
 -167.7  ***
 -189.4  ***

 -141.9  ***  total noise at 50 Hz

 3.6-09 GSBD  e_n @ 100 Hz.
  100.0 GSBE  frequency

   -1.0  ***

 -188.9  ***
 -198.9  ***
 -145.9  ***
 -168.9  ***
 -193.4  ***

 -145.9  ***  total noise at 100 Hz

 6.1-16 GSBC  i_n @ 200 Hz
 3.2-05 GSBD  e_n @ 200 Hz
  200.0 GSBE  frequency

   -0.3  ***

 -168.2  ***
 -198.2  ***
 -151.2  ***
 -169.9  ***
 -198.6  ***

 -151.1  ***  total noise at 200 Hz

 6.2-16 GSBC  i_n @ 500 Hz
 2.8-09 GSBD  e_n @ 500 Hz
  500.0 GSBE  frequency

    0.0  ***

 -188.0  ***
 -198.0  ***
 -158.9  ***
 -171.1  ***
 -206.2  ***

 -158.7  ***  total noise at 500 Hz.

 6.3-16 GSBC  i_n @ 1000 Hz
 2.7-09 GSBD  e_n @ 1000 Hz
 1000.0 GSBE  frequency

    0.0  ***

 -187.9  ***
 -197.9  ***
 -164.9  ***
 -171.4  ***
 -212.0  ***

 -164.0  ***  total noise at 1000 Hz
```

Example 1-6.1 continued

This example points up one of the problems associated with using the characteristics of the sensor impedance along with the amplifier input resistance to effect frequency shaping. It will be noticed that the dominant source of noise comes from the thermal noise of the input resistor. The low noise characteristics of the input transistor are buried by the input resistor noise contribution.

If the input resistor is made larger, the noise contribution of the input resistor will be less. Although this statement may seem backwards, the logic may be seen by looking at the input resistor and its noise generator as a Norton equivalent source instead of a Thevenin equivalent as is presently used. In this light, one can see that the injected noise current is proportional to $1/\sqrt{R}$. Since other circuit impedances are unchanged, lower injected noise current means lower input resistor noise contribution.

The input resistor noise contribution may also be reduced by lowering the sensor impedance to lower the noise voltage resulting from the input resistor noise current.

To illustrate the above point, the example is rerun using a larger input resistor; 100 megohms is used instead of 796 kilohms. The HP-97 printout for this case is shown on the next page. The noise contribution of the input resistor loses dominance above 500 Hz in this case.

Fortunately, the ocean self noise is greatest at low frequencies, and low noise performance is less critical here.

EXAMPLE 1-6.1 CONTINUED

PROGRAM INPUT

100.+06	STO7	store new R_{in}
	PREG	print registers to show currently stored values
10.00+00	0	sensor resistance
4.000-09	1	sensor capacitance
0.000+00	2	sensor inductance
0.000+00	3	primary resistance
100.0+48	4	primary inductance
0.000+00	5	secondary capacitance
1.000+00	6	secondary resistance
100.0+06	7	input resistance
1.000+00	8	xfmr turns ratio

PROGRAM OUTPUT

6.-16	GSBC	$\overline{i_n}$ @ 10 Hz
7.-09	GSBD	$\overline{e_n}$ @ 10 Hz
10.	GSBE	frequency
0.0	***	A_v, network gain, dB
-187.9	***	R_s+R_{pri} thermal noise, dBV
-197.9	***	R_{sec} thermal noise, dBV
-145.9	***	R_{in} thermal noise, dBV
-163.1	***	$\overline{e_n}$, transistor, dBV
-172.4	***	$\overline{i_n}$*Z equiv. noise, dBV
-145.6	***	total noise (RSS), dBV

5.3-09	GSBD	$\overline{e_n}$ @ 20 Hz *
20.0	GSBE	frequency
0.0	***	
-187.9	***	
-197.9	***	
-151.9	***	
-165.5	***	
-178.5	***	
-151.7	***	total noise at 20 Hz

*$\overline{i_n}$ is unchanged from the last entry.

4.1-09	GSBD	$\overline{e_n}$ @ 50 Hz
50.0	GSBE	frequency
0.0	***	
-187.9	***	
-197.9	***	
-159.5	***	
-167.7	***	
-166.4	***	
-159.2	***	total noise at 50 Hz

3.6-09	GSBD	$\overline{e_n}$ @ 100 Hz
100.0	GSBE	frequency
0.0	***	
-167.9	***	
-197.9	***	
-165.9	***	
-168.9	***	
-192.4	***	
-164.1	***	total noise at 100 Hz.

6.1-16	GSBC	$\overline{i_n}$ @ 200 Hz
3.2-09	GSBD	$\overline{e_n}$ @ 200 Hz
200.0	GSBE	frequency
0.0	***	
-187.9	***	
-197.9	***	
-171.9	***	
-169.9	***	
-198.3	***	
-167.7	***	total noise at 200 Hz

6.2-16	GSBC	$\overline{i_n}$ @ 500 Hz
2.8-09	GSBD	$\overline{e_n}$ @ 500 Hz
500.0	GSBE	frequency
0.0	***	
-187.9	***	
-197.9	***	
-179.9	***	
-171.1	***	
-206.1	***	
-170.4	***	total noise at 500 Hz

6.3-16	GSBC	$\overline{i_n}$ @ 1000 Hz
2.7-09	GSBD	$\overline{e_n}$ @ 1000 Hz
1000.0	GSBE	frequency
0.0	***	
-187.9	***	
-197.9	***	
-165.9	***	
-171.4	***	
-212.0	***	
-171.1	***	total noise at 1000 Hz

Example 1-6.2

A small hydrophone is to be matched to a low-noise preamplifier for optimum noise performance at 30 kHz. The hydrophone equivalent circuit is shown in Fig. 1-6.2. The amplifier input transistor will be a 2N4867A FET operating at a drain current of I_{dss}.

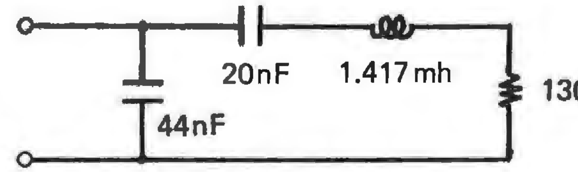


Figure 1-6.2 Hydrophone equivalent circuit.

Table 1-6.2 Current and voltage noise densities of 2N4867A operating at I_{dss}.

Frequency, kHz	$\overline{i_n}$, A/$\sqrt{Hz}$	$\overline{e_n}$, V/$\sqrt{Hz}$
10	8.0×10^{-16}	2.3×10^{-9}
15	1.0×10^{-15}	
20	1.2×10^{-15}	
25	1.4×10^{-15}	
30	1.6×10^{-15}	2.3×10^{-9}
35	1.75×10^{-15}	2.2×10^{-9}
40	1.9×10^{-15}	
45	2.15×10^{-15}	
50	2.4×10^{-15}	
55	2.7×10^{-15}	
60	3.0×10^{-15}	2.2×10^{-9}

Before the analysis is started, the transformer turns ratio, primary inductance, and amplifier input resistance must be chosen. The transformer ratio should be kept low to minimize the current noise contribution of the input transistor.

The parallel equivalent circuit of the hydrophone at 30 kHz is required. The capacitive part will be resonated by the transformer primary inductance, leaving only the resistive part. Figure 1-6.3 shows the parallel equivalent circuit before resonating, and Fig. 1-6.4 shows the HP-97 calculations used to obtain the parallel equivalent circuit.

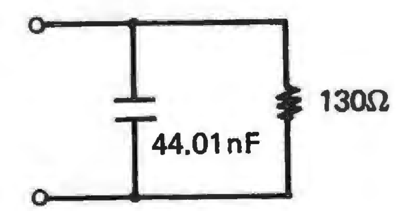


Figure 1-6.3 Parallel equivalent circuit of hydrophone at 30 kHz.

30000.	PI	enter frequency and
	x	calculate and store
2.	x	$\omega = 2\pi f \rightarrow R_E$
	STOE	
1.407-03	x	form ωL
	RCLE	
20000.-12	x	form $1/(\omega C)$
	1/X	
	-	form and print
-44.95-03	***	$\omega L - 1/(\omega C) = \text{Im } Z$
130.	→P	load Re Z
	1/X	
	X⇄Y	convert impedance
	CHS	to admittance
	X⇄Y	
	→R	
	1/X	$\frac{1}{\text{Re } Y}$ in ohms
130.0+00	***	
	1/X	Re Y back in mhos
	X⇄Y	Im Y
44000.-12	RCLE	add clamp capacity
	x	susceptance
	+	
	RCLE	convert total
	÷	susceptance to
44.01-09	***	capacitance & print

Figure 1-6.4 HP-97 printout showing calculations used to find the parallel equivalent circuit at 30 kHz.

The thermal noise of the equivalent parallel resistor in a one Hz band is:

$$\overline{e_n}\ (130\ \Omega) = \sqrt{4KT(130)} = 1.45 \times 10^{-9}\ \text{V}/\sqrt{\text{Hz}}$$

If the transformer raises this noise to 6 dB above the transistor noise, the RSS sum of both resistor and transistor noises will be 1 dB higher than the resistor noise alone. The transformer turns-ratio necessary to meet this condition is:

$$n = \frac{2(2.2 \times 10^{-9})}{1.45 \times 10^{-9}} = 3.03$$

The noise current contribution to the total noise voltage also may be calculated (only Re Z_{in} is used as Im Z_{in} is resonated out):

$$\overline{e_n} = \overline{i_n} \cdot n^2 \cdot |Z_{in}| = (1.6 \times 10^{-15})(10^2)(130) = 20.8 \times 10^{-12}\ \mathrm{V}/\sqrt{\mathrm{Hz}}$$

This contribution is insignificant compared to the voltage noise term, and the transformer ratio may be raised to make the dominant noise source that of the hydrophone resistance only. This will be the best noise performance obtainable.

With a transformer ratio of 10:1, the equivalent hydrophone resistor noise is 1.45×10^{-8} V/$\sqrt{\mathrm{Hz}}$ at the transistor input, and the RSS of both the transistor and resistor noises is 1.467×10^{-8}. This RSS voltage is only 0.1 dB above the resistor noise alone!

To represent the equivalent hydrophone shunt capacity (44.01 nF), the transformer secondary capacitance term, C_{sec} is used. This equivalent secondary capacity is the primary capacity (hydrophone capacity) divided by the square of the turns ratio:

$$C_{sec} = (44.01\ \mathrm{nF})/(10^2) = 440\ \mathrm{pF}$$

The primary inductance is chosen to parallel resonate with the equivalent hydrophone capacity, 44.01 nF, at the design frequency of 30 kHz. This primary inductance is:

$$L_{pri} = 1/((2\pi f)^2 C) = 1/((2\pi 30000)^2 \cdot 44.01 \times 10^{-9})$$

$$L_{pri} = 639.5\ \mu\mathrm{h}$$

The "Q" of the network is $R/(2\pi fL) = 1.078$, which means the approximate bandwidth of the network is 30000/1.078 = 27829 Hz. Additional broadbanding using the shunting effect of an amplifier input resistor is not necessary. This input resistor may be removed altogether as the transformer secondary provides the dc return for the transistor gate connection. The input resistor will be omitted by making its value 10^{50} ohms.

The HP-97 printout for this example is shown on the next page, and the equivalent circuit is shown in Fig. 1-6.5.

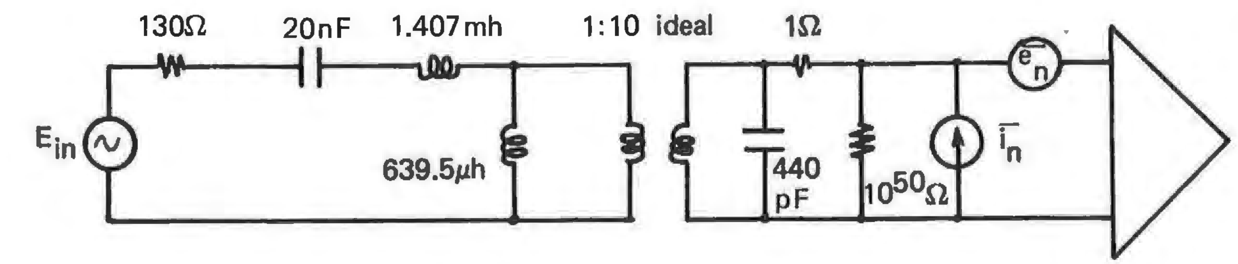


Figure 1-6.5 Equivalent circuit for hydrophone and amplifier.

HP-97 PRINTOUT FOR EXAMPLE 1-6.2

130.0	STO0	R_s
20000.-12	STO1	C_s
1.407-03	STO2	L_s
0.0	STO3	R_{pri}
639.5-06	STO4	L_{pri}
440.-12	STO5	C_{sec}
1.0	STO6	R_{sec}
1.+50	STO7	R_{in}
10.0	STO8	$n_{transformer}$
	GSBA	select dB mode for p/o

PROGRAM OUTPUT

8.-16	GSBC	load $\bar{I}_n$@10kHz
2.3-09	GSBD	load $\bar{e}_n$@ "
10000.0	GSBE	load freq & start
-3.5	***	A_v dB
-186.3	***	
-197.9	***	
-624.3	***	
-172.8	***	
-228.3	***	
-172.0	***	RSS of all noise $\bar{e}$'s, dBV
1.-15	GSBC	$\bar{I}_n$ @ 15 kHz *
15000.0	GSBE	freq & start (*$\bar{e}_n$ unchged)
7.4	***	
-169.4	***	
-197.9	***	
-618.1	***	
-172.8	***	
-220.2	***	
-167.7	***	$\bar{e}_n$ tot @ 15kHz
1.2-15	GSBC	$\bar{I}_n$ @ 20 kHz
20000.0	GSBE	freq & start
19.6	***	
-157.1	***	
-197.9	***	
-610.1	***	
-172.8	***	
-210.6	***	
-157.0	***	$\bar{e}_n$ tot @ 20kHz
1.4-15	GSBC	$\bar{I}_n$ @ 25 kHz
25000.0	GSBE	freq & start
21.9	***	
-154.9	***	
-197.9	***	
-611.8	***	
-172.8	***	
-211.0	***	
-154.8	***	$\bar{e}_n$ tot @ 25kHz
1.6-15	GSBC	$\bar{I}_n$ @ 30 kHz
30000.0	GSBE	freq & start
20.0	***	
-156.8	***	
-197.9	***	
-615.6	***	
-172.8	***	
-213.6	***	
-156.7	***	$\bar{e}_n$ tot @ 30kHz
1.75-15	GSBC	$\bar{I}_n$ @ 35 kHz
2.2-09	GSBD	$\bar{e}_n$ @ "
35000.0	GSBE	freq & start
21.3	***	
-155.4	***	
-197.9	***	
-612.8	***	
-173.2	***	
-210.1	***	
-155.4	***	$\bar{e}_n$ tot @ 35kHz
1.9-15	GSBC	$\bar{I}_n$ @ 40 kHz
40000.0	GSBE	freq & start
23.4	***	
-153.4	***	
-197.9	***	
-608.4	***	
-173.2	***	
-204.9	***	
-153.3	***	$\bar{e}_n$ tot @ 40kHz
2.15-15	GSBC	$\bar{I}_n$ @ 45 kHz
45000.0	GSBE	freq & start
19.6	***	
-157.1	***	
-197.9	***	
-610.1	***	
-173.2	***	
-205.5	***	
-157.0	***	$\bar{e}_n$ tot @ 45kHz
2.4-15	GSBC	$\bar{I}_n$ @ 50 kHz
50000.0	GSBE	freq & start
14.5	***	
-162.3	***	
-197.9	***	
-613.6	***	
-173.2	***	
-208.1	***	
-162.0	***	$\bar{e}_n$ tot @ 50kHz
2.7-15	GSBC	$\bar{I}_n$ @ 55 kHz
55000.0	GSBE	freq & start
10.5	***	
-166.3	***	
-197.9	***	
-616.2	***	
-173.2	***	
-209.6	***	
-165.5	***	$\bar{e}_n$ tot @ 55kHz
3.-15	GSBC	$\bar{I}_n$ @ 60 kHz
60000.0	GSBE	freq & start
7.4	***	
-169.4	***	
-197.9	***	
-618.1	***	
-173.2	***	
-210.6	***	
-167.8	***	$\bar{e}_n$ tot @ 60kHz

Example 1-6.2 is meant to illustrate both the program functioning and to give some insight on hydrophone matching. The gain versus frequency response has two peaks, which is characteristic of doubly tuned networks.

The whole subject of optimum hydrophone matching is beyond the scope of this program and discussion. Equiripple passband response and optimum noise performance may be simultaneously obtained with higher order matching networks which represent bandpass filter like structures and include the hydrophone equivalent circuit in the filter structure. Typical broadbanding networks are fifth order and have Chebyshev responses. These networks are an extension of the work of Fano [23] and Matthaei [37].

Derivation of Equations Used

The network shown in Fig. 1-6.1 is redrawn with the components on the secondary side of the transformer reflected to the primary side, and the thermal noise sources of the resistors added. This new network is shown in Fig. 1-6.6.

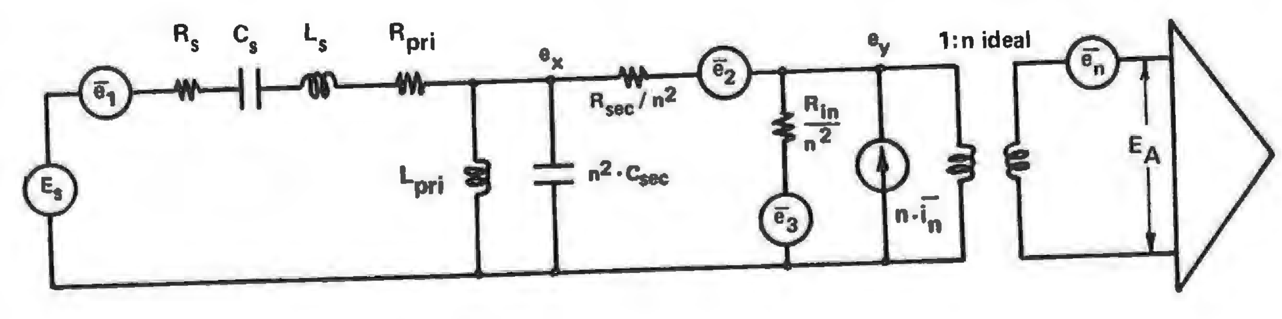


Fig. 1-6.6 Network of Fig. 1-6.1 redrawn with the transformer moved to the right side.

The network of Fig. 1-6.6 is shown in Fig. 1-6.7 with the individual element groups replaced by generalized admittance blocks. The noise voltage densities of the noise generators are defined by Eqs. (1-6.1) through (1-6.3), and the admittance blocks are defined by Eqs. (1-6.4) through (1-6.7).

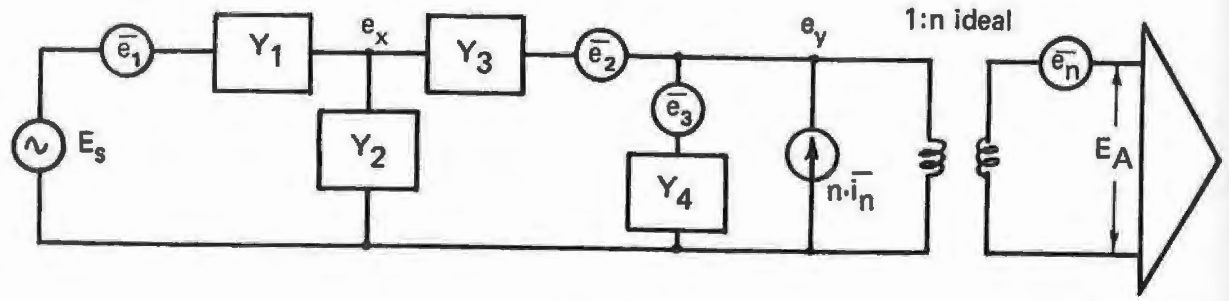


Figure 1-6.7 Network of Fig. 1-6.6 redrawn with generalized admittance blocks.

$$\bar{e}_1 = \sqrt{4KT(R_s + R_{pri})} \tag{1-6.1}$$

$$\bar{e}_2 = (1/n)\sqrt{4KTR_{sec}} \tag{1-6.2}$$

$$\bar{e}_3 = (1/n)\sqrt{4KTR_{in}} \tag{1-6.3}$$

$$Y_1 = \frac{1}{R_s + R_{pri} + sL_s + 1/(sC_s)} = \frac{1}{R_s + R_{pri} + j(\omega L_s - 1/(\omega C_s))} \tag{1-6.4}$$

$$Y_2 = s(n^2 \cdot C_{sec}) + 1/(sL_{pri}) = j(n^2\omega C_{sec} - 1/(\omega L_{pri})) \tag{1-6.5}$$

$$Y_3 = n^2/R_{sec} \qquad s = j\omega \tag{1-6.6}$$

$$Y_4 = n^2/R_{in}$$

Where K is Boltzmann's constant (1.380 x 10^{-23} Joules/K), and T is the temperature in Kelvin (290 K at room temperature).

The nodal equations are written from Fig. 1-6.7:

$$\begin{bmatrix} (Y_1 + Y_2 + Y_3) & (-Y_3) \\ (-Y_3) & (Y_3 + Y_4) \end{bmatrix} \cdot \begin{bmatrix} e_x \\ e_y \end{bmatrix} = \begin{bmatrix} Y_1(\bar{e}_1 + e_s) - Y_3\bar{e}_2 \\ Y_3\bar{e}_2 + Y_4\bar{e}_3 + n\cdot\bar{I}_n \end{bmatrix} \tag{1-6.8}$$

The variable, e_y, is obtained using Cramer's rule. The determinant of the coefficient matrix is designated Δ.

$$\Delta = (Y_1 + Y_2 + Y_3)(Y_3 + Y_4) - Y_3^2$$

which upon rearranging yields:

$$\Delta = (Y_1 + Y_2 + Y_3)(Y_4) + (Y_1 + Y_2)(Y_3) \tag{1-6.9}$$

Substituting the constant matrix (right hand side) into the second column of the coefficient matrix, and evaluating the determinant yields the following:

$$n\cdot e_y = (n/\Delta)\left[(Y_1+Y_2+Y_3)(Y_3\bar{e}_2+Y_4\bar{e}_3+n\cdot\bar{i}_n)+(Y_1Y_3)(\bar{e}_1+E_s)-(Y_3^2)(\bar{e}_2)\right] \tag{1-6.10}$$

Simplifying and removing term subtraction yields:

$$n\cdot e_y = (n/\Delta)\left[(Y_1+Y_2+Y_3)(Y_4\bar{e}_3+n\cdot\bar{i}_n)+(Y_1Y_3)(\bar{e}_1+E_s)+(Y_1-Y_2)(Y_3\bar{e}_2)\right] \tag{1-6.11}$$

The voltage gain of the network is: $n\frac{\partial e_y}{\partial E_s} = \frac{\partial e_A}{\partial E_s} = A_v$, or: (1-6.12)

$$A_v = (nY_1Y_3)/(\Delta), \tag{1-6.13}$$

In terms of magnitude only:

$$A_v = n\cdot|Y_1|\cdot|Y_3|/|\Delta| \tag{1-6.14}$$

Since the noise voltages $\bar{e}_2$, $\bar{e}_3$, and $\bar{e}_n$, and the current $\bar{i}_n$ are random in nature, their addition must be done in RSS fashion to obtain the overall RMS noise voltage at the amplifier input, e_A, i.e.,

$$\bar{e}_A^{\,2} = \bar{e}_n^{\,2} + n^2\cdot\bar{e}_y^{\,2} \tag{1-6.15}$$

Upon expanding:

$$\bar{e}_A^{\,2} = \bar{e}_n^{\,2} + \frac{n^2}{|\Delta|^2}\left(|Y_1+Y_2+Y_3|^2\cdot(\bar{e}_3^{\,2}|Y_4|^2+n^2\bar{i}_n^{\,2}) + |Y_1Y_3|^2\cdot\bar{e}_1^{\,2} + |Y_1+Y_2|^2 Y_3^2\,\bar{e}_2^{\,2}\right) \tag{1-6.16}$$

This program uses Eqs. (1-6.14) and (1-6.16) to calculate the overall noise voltage density.

Program Listing I

```
001  *LBLA   SELECT OUTPUT IN dB & dBV
002    CF1
003    RTN
004  *LBLB   SELECT OUTPUT IN RATIO
005    SF1   AND VOLTS
006    RTN
007  *LBLC   LOAD AMPLIFIER INPUT CURRENT
008    P⇄S   NOISE DENSITY IN A/√Hz
009    F3?   if numeric entry, jump to
010   GTO0   storage routine
011   RCL0   recall presently stored value
012    P⇄S   jump to print and space
013   GTO2   routine
014  *LBL0   store entered value of īn,
015   STO0   and return control to
016    P⇄S   keyboard
017    RTN
018  *LBLD   LOAD AMPLIFIER INPUT VOLTAGE
019    F3?   NOISE DENSITY IN V/√Hz
020   GTO0   if numeric entry, jump
021   RCL9   recall presently stored value
022  *LBL2   print and space routine
023   PRTX
024    SPC
025    RTN   return control to keyboard
026  *LBL0   store entered value of ēn
027   STO9
028    RTN   return control to keyboard
029  *LBLE   LOAD ANALYSIS FREQ & START
030    F3?   if numeric entry, store it
031   STOI
032   RCLI   recall present stored freq
033   GSB3   if flag 0, space
034   ENT↑
035      +   form and store ω=2πf
036     Pi
037      x
038   STOE
039   RCL2   form ωLsens
040      x
041   RCLE
042   RCL1   form 1/(ωCsens)
043      x
044    1/X
045      -   Im Z1 = ωLs - 1/(ωCs)
046   RCL0
047   RCL3   Re Z1 = Rs + Rpri
048      +
049     →P   convert rectangular to polar
050    1/X   form and store |Y1|
051   STOD
052    X⇄Y
053    CHS   finish complex inverse,
054    X⇄Y   and return output in
055     →R   rectangular co-ordinates
056   STOA   store Re(Y1 + Y2)  Re Y1
057    X⇄Y   calculate and store
058   RCL5   Im(Y1 + Y2):
059   RCL8
060     X²   calculate n²ωCsec
061      x
062   RCLE
063      x
064      +
065   RCLE   calculate 1/(ωLpri)
066   RCL4
067      x
068    1/X
069      -
070   STOB   form and store Im(Y1 + Y2)
071   RCLA   take Y1 + Y2 to polar
072     →P
073   RCL8   calculate and store:
074     X²   Y3 = n²/Rsec
075   RCL6
076      ÷
077   STOC
078      x   form and store |Y3|·|Y1 + Y2|
079    P⇄S
080   STO2
081    P⇄S
082   RCLB   Im(Y1+Y2+Y3)  Im(Y1+Y2)
083   RCLA   calculate Re(Y1+Y2+Y3)
084   RCLC
085      +
086     →P   form and store Y1+Y2+Y3
087   STOE
088   RCL8   form |Y4|·|Y1+Y2+Y3|
089     X²
090      x
091   RCL7
092      ÷
093     →R   form Re&Im(Y4(Y1+Y2+Y3))
094   RCLA   form Re(Y3(Y1+Y2))
095   RCLC
096      x
097      +   form Re Δ
098    X⇄Y   form Im(Y3(Y1+Y2))
099   RCLB
100   RCLC
101      x
102      +   form Im Δ
103     →P   form |Δ|
104   RCL8   form and store n/|Δ|
105      ÷
106    1/X
107   STOA
108   RCLD   form |Y1Y3|
109   RCLC
110      x
```

REGISTERS									
0 R_s	1 C_s	2 L_s	3 R_{pri}	4 L_{pri}	5 C_{sec}	6 R_{sec}	7 R_{in}	8 n	9 $\bar{e}_{n_{amp}}$
S0 $\bar{I}_{n_{amp}}$	S1 $\sum V^2$	S2 $\lvert Y_3(Y_1+Y_2)\rvert$	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
Re Y_1, $\frac{n}{\lvert\Delta\rvert}$	$Im(Y_1+Y_2)$, 4KT	$\lvert Y_3\rvert$, $\lvert Y_1Y_3\rvert$	$\lvert Y_1\rvert$, $\lvert A_v\rvert$	$2\pi f$, $\lvert Y_1+Y_2+Y_3\rvert$	f, the freq for analysis

Program Listing II

```
111   STOC   calculate and store:
112      x   |Av| = n·|Y1Y3|/|Δ|
113   STOD
114   GSB1   print Av or 20·log Av
115      0   initialize Σ V² register
116    P⇄S
117   STO1
118    P⇄S
119   GSB3   space if flag 0 is set
120      1   form and store 4KT
121      .
122      6
123      1
124      7
125      3
126      6
127    EEX
128    CHS
129      2
130      0
131   STOB
132   RCL0   calculate and output:
133   RCL3   Av·√(4KT(Rs + Rpri))
134      +   which is the transformer
135      x   primary resistance and
136     √X   sensor resistance thermal
137   RCLD   voltage noise density
138      x
139   GSB1
140   RCLB   calculate and output:
141   RCL6   |Y3(Y1+Y2)/Δ|·√(4KTRsec)
142      x   which is the transformer
143     √X   secondary resistance thermal
144   RCL8   voltage noise density
145      ÷
146    P⇄S
147   RCL2
148    P⇄S
149      x
150   RCLA
151      x
152   GSB1
153   RCLB   calculate and output:
154   RCL7
155      ÷   n/|Δ|·|Y4(Y1+Y2+Y3)|·√(4KTRin)
156     √X   which is the thermal noise
157   RCL8   voltage density due to the
158      x   amplifier input resistance
159   RCLA
160   RCLE
161      x
162   STOE
163      x
164   GSB1
165   RCL9   recall and output the
             amplifier noise voltage dens
166   GSB1
167    P⇄S   calculate and output the
168   RCL0   voltage noise density
169    P⇄S   caused by the amplifier
170   RCL8   input current noise density
171      x   acting on the equivalent
172   RCLE   circuit impedance
173      x
174   GSB1
175   GSB3   space if flag 0 is set
176    P⇄S
177   RCL1   recall and output the RSS
178    P⇄S   of all the above noise
179     √X   voltage densities
180   GSB1   ( √ΣV² )
181   GSB3
182   GTO3
183  *LBL1   output subroutine:
184     X²   store ΣV²
185    P⇄S
186   ST+1
187    P⇄S
188   LSTX   recall V
189    F1?   if flag 1, output voltage
190   GTO1   in engineering format
191    FIX   flag 1 is cleared: output
192   DSP1   20 log V in fix 1 format
193    LOG
194      2
195      0
196      x
197    RND
198   GTO2
199  *LBL1
200    ENG
201   DSP3
202  *LBL2   print-R/S subroutine
203    F0?
204   PRTX
205    F0?
206    RTN
207    R/S
208    RTN
209  *LBL3   space if flag 0 subroutine
210    F0?
211    SPC
212    RTN
213  *LBLa   print - R/S toggle
214    CF0
215      0   a "0" displayed indicates
216    RTN   R/S mode selected
217  *LBLa
218    SF0
219      1   a "1" displayed indicates
220    RTN   print mode selected
```

LABELS				
A select dB	B select linear	C load $\bar{I}_n$	D load $\bar{e}_n$	E input freq & go
a	b	c	d	e
0 local lbl	1 output	2 print R/S	3 spc if F0	4
5	6	7	8	9

FLAGS
0 R/S, prt
1 dB/ linear
2
3 data entry

SET STATUS				
FLAGS	ON	OFF	TRIG	DISP
0	■		DEG ■	FIX
1		■	GRAD	SCI
2			RAD	ENG
3				n____

Part 2

FILTER DESIGN

PROGRAM 2-1 BUTTERWORTH AND CHEBYSHEV FILTER ORDER CALCULATION.

Program Description and Equations Used

This program calculates the minimum filter order required to meet specifications for maximum passband attenuation (Ap_{dB}) and minimum stopband attenuation (As_{dB}) for the Butterworth or Chebyshev filter approximations. A second part of the program calculates the stopband-to-passband frequency ratio, λ, if the filter order and type are given. Furthermore, a third part of the program predicts the stopband attenuation if n, λ, Ap_{dB}, and the filter order are provided.

Figures 2-1.1 and 2-1.2 are nomographs adapted from Kawakami [34], and can prove useful to rough out the problem and provide tradeoffs. Once the desired parameters have been estimated, this program may be used to fine-tune the results.

Equation (2.1.1) is the analytic expression for the Butterworth amplitude response characteristic.

$$A_s^2 - 1 = (A_p^2 - 1)\ \lambda^{2n} \qquad (2\text{-}1.1)$$

where

$$A_s^2 = 10^{0.1As_{dB}} \qquad (2\text{-}1.2)$$

and

$$A_p^2 = 10^{0.1Ap_{dB}} \qquad (2\text{-}1.3)$$

The quantities A_s and A_p are ratios greater than one (it is the convention to express attenuation as positive decibels).

Equations (2-1.1), (2-1.2), and (2-1.3) can be used to find expressions for As_{dB}, λ, or n:

$$As_{dB} = 10 \cdot \log \left[(A_p^2 - 1)\lambda^{2n} + 1\right] \quad (2.1\text{-}4)$$

$$\lambda = \left[\frac{A_s^2 - 1}{A_p^2 - 1}\right]^{\frac{1}{2n}} = \left[\sqrt{\frac{A_s^2 - 1}{A_p^2 - 1}}\right]^{\frac{1}{n}} \quad (2\text{-}1.5)$$

$$n = \frac{\ln \sqrt{\dfrac{A_s^2 - 1}{A_p^2 - 1}}}{\ln \lambda} \quad (2\text{-}1.6)$$

Equation (2-1.7) is the analytic expression for the Chebyshev amplitude characteristic where A_s^2 and A_p^2 are defined by Eqs. (2-1.2) and (2-1.3). Equation (2-1.7) can also yield expressions for As_{dB}, λ, or n:

$$A_s^2 - 1 = (A_p^2 - 1)\left[\cosh (n \cosh^{-1}\lambda)\right]^2 \quad (2\text{-}1.7)$$

$$As_{dB} = 10 \cdot \log \left[(A_p^2 - 1)(\cosh (n \cdot \cosh^{-1}\lambda)^2 + 1\right]^2 \quad (2\text{-}1.8)$$

$$\lambda = \cosh \left(\frac{1}{n} \cosh^{-1} \sqrt{\frac{A_s^2 - 1}{A_p^2 - 1}}\right) \quad (2\text{-}1.9)$$

$$n = \frac{\cosh^{-1} \sqrt{\dfrac{A_s^2 - 1}{A_p^2 - 1}}}{\cosh^{-1} \lambda} \quad (2\text{-}1.10)$$

A certain degree of similarity can be noticed between the Butterworth and Chebyshev equations. Keeping in mind that ln and exp are complementary operations as are cosh and $\cosh^{-1}$, and noticing that y^x can be expressed as exp (x · ln y), then replacing ln with cosh and exp with $\cosh^{-1}$ will convert the Butterworth formulas to the Chebyshev

formulas. This technique is used by this program where flag 1 indicates the function to be used (set for Butterworth).

A separate subprogram is also included to aid in the specification of bandpass or bandstop filters. The characteristics of these filters are symmetrical when plotted on logarithmic frequency scales (log paper). This characteristic implies geometric symmetry of the various defining frequencies (-3dB, etc.) about the filter center frequency, i.e., the center frequency is the square root of the product of similar response frequencies located above and below the center frequency.

To use the bandstop and bandpass programs in this section, the filter center frequency (f_o) and bandwidth (BW) are needed, however, when specifying the filter initially, the bandedge frequencies may be of greater interest. The separate subprogram provides the conversion between center frequency and bandwidth, and upper and lower bandedge frequencies (f_{upr} and f_{lwr}), and vice-versa. The definition of "bandedge frequencies" in the present context means a pair of frequencies (one on either side of the center frequency) where the filter attenuation is the same, i.e., -0.01 dB, -3 dB, -60 dB, etc.

To convert from center frequency and bandwidth to upper and lower bandedge frequencies, Eqs. (2-1.11) and (2-1.12) apply.

$$f_{upr} = (BW/2) + \sqrt{(BW/2)^2 + f_o^2} \quad (2\text{-}1.11)$$

$$f_{lwr} = (f_o^2)/f_{upr} \quad (2\text{-}1.12)$$

To do the reverse conversion, i.e., to go from upper and lower bandedge frequencies to center frequency and bandwidth, Eqs. (2-1.13) and (2-1.14) apply.

$$f_o = \sqrt{(f_{upr})(f_{lwr})} \quad (2\text{-}1.13)$$

$$BW = f_{upr} - f_{lwr} \quad (2\text{-}1.14)$$

In the case of a bandpass or bandstop filter, the stopband-to-passband frequency ratio, λ, still holds. The user should remember to use bandwidths, and not bandedge frequencies. This is an easy trap to fall into since bandedge frequencies and bandwidths can be one and the same for lowpass filters.

2-1 User Instructions

BUTTERWORTH AND CHEBYSHEV FILTER ORDER CALCULATION				
Butter-worth	Chebyshev	print, R/S	BW↑f_o → f_{upr},f_{lwr}	f_{upr}↑f_{lwr} → f_o, BW
Ap_{dB}	As_{dB}	n→λ	λ→n	λ→ As_{dB}

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load program card (either side)			
2	Select print (HP-97), or R/S (HP-67) option		f C f C f C	1 (print) 0 (R/S) 1 ⋮
3	Select filter type: Butterworth Chebyshev		 f A f B	
4	Load the maximum passband attenuation in dB	Ap_{dB}	A	
5	Load the minimum stopband attenuation in dB	As_{dB}	B	
6	To find filter order, n, given the frequency ratio, λ, load λ	λ	D	n
7	To find the frequency ratio, λ, given the filter order, n: load n (n must be integer)	n	C	λ
8	After finding n, to find As (dB) given λ a) perform step 7 to store n b) load λ Step 8b may be repeated with other values of λ without having to repeat step 8a.	 λ	 E	 As_{dB}
9	A separate program section to aid with bandpass filter selection, enter bandwidth and center frequency and calculate the upper and lower bandedge frequencies, or vice-versa load bandwidth (for any dB down points) load center frequency load upper bandedge frequency load lower bandedge frequency	 BW, Hz f_o, Hz f_{upr}, Hz f_{lwr}, Hz	 ENT↑ f D ENT f E	 f_{upr}, Hz f_{lwr}, Hz f_o, Hz BW, Hz

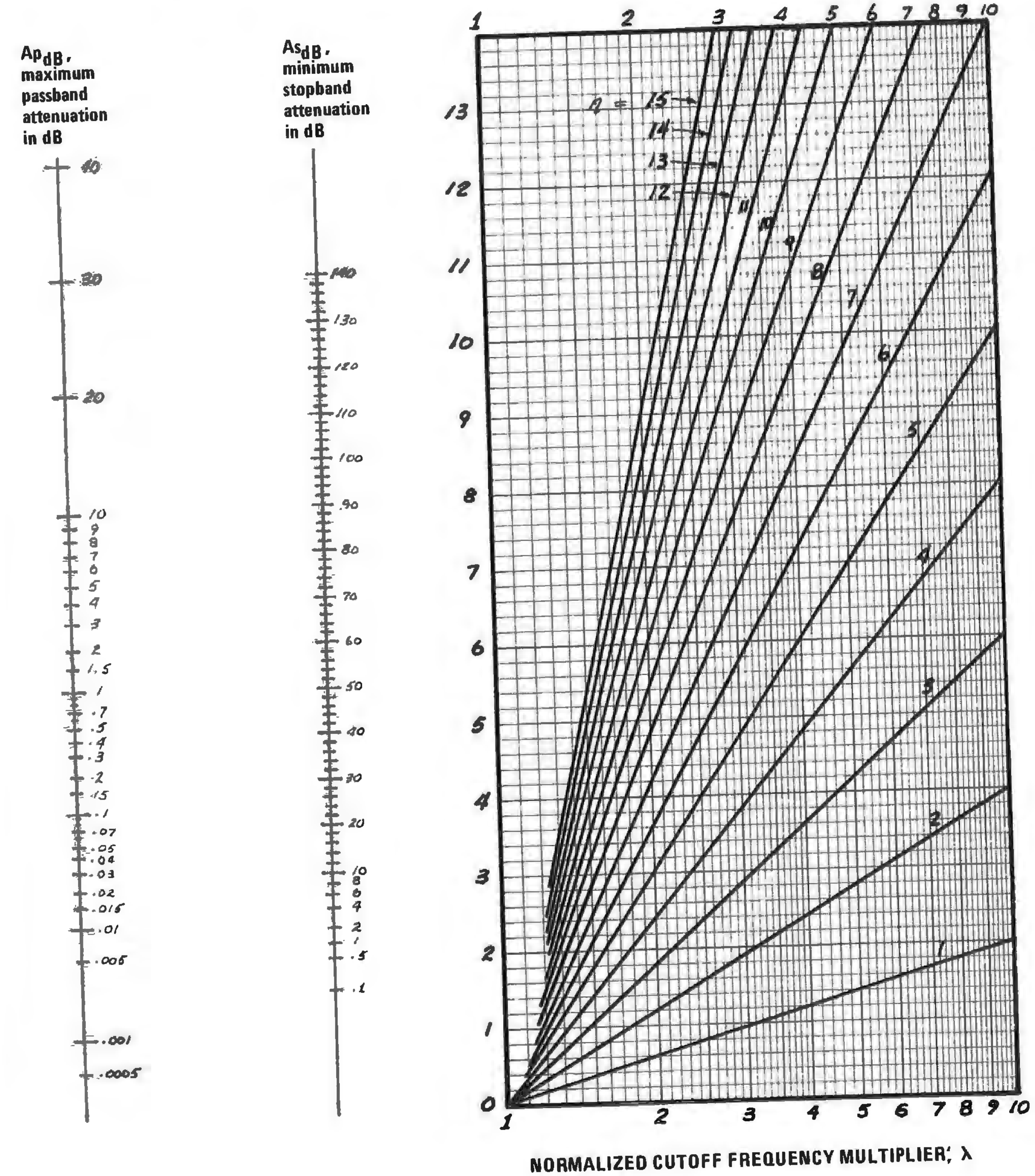


Figure 2-1.1 Butterworth filter nomograph.

$$A_s^2 - 1 = (A_p^2 - 1)\lambda^{2n}.$$

Adapted from Kawakami [34]

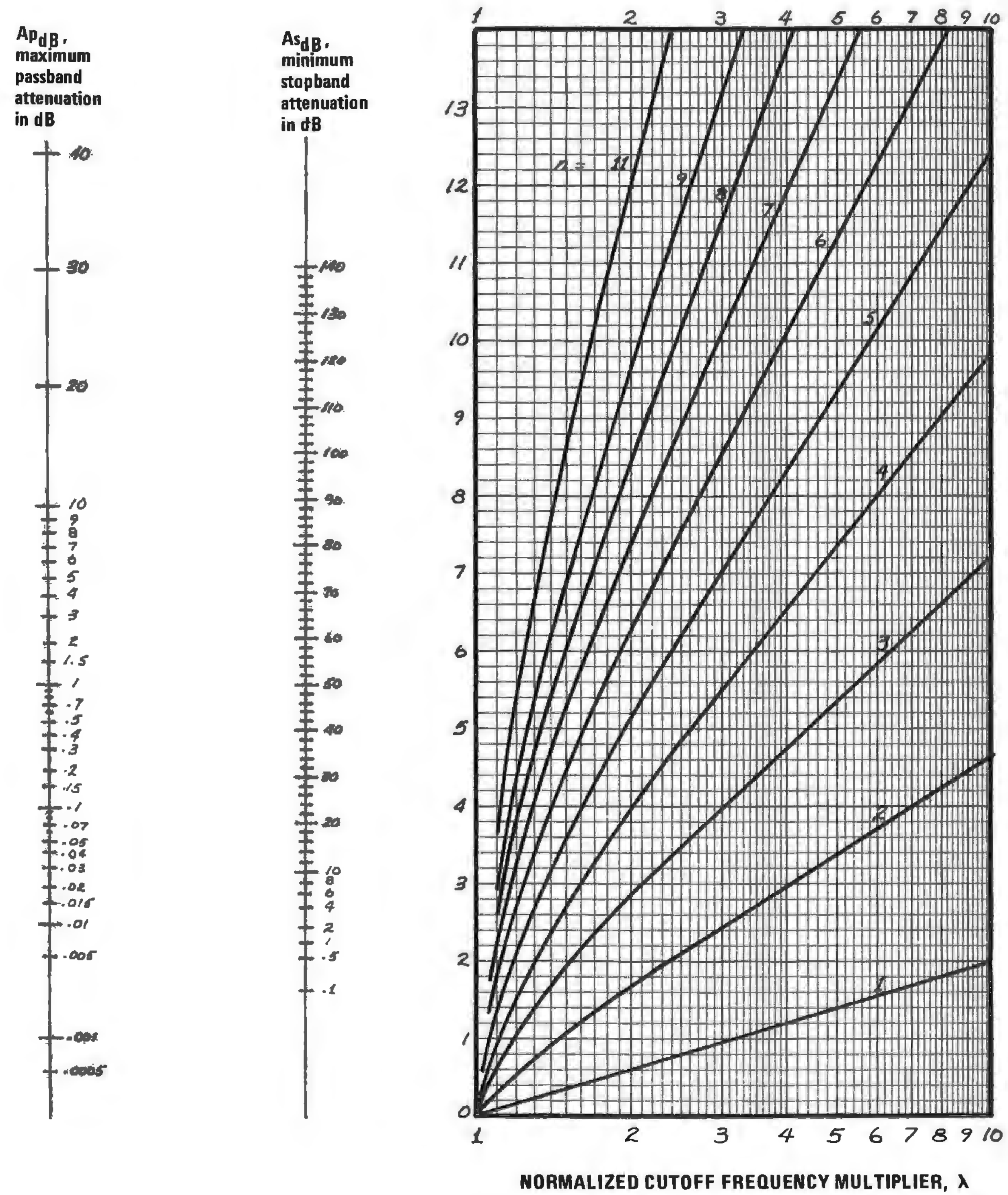


Figure 2-1.2 Chebyshev filter nomograph.

$$A_s^2 - 1 = (A_p^2 - 1)\{\cosh(n \cdot \cosh^{-1}\lambda)\}$$

Adapted from Kawakami [34]

How to Use the Nomographs

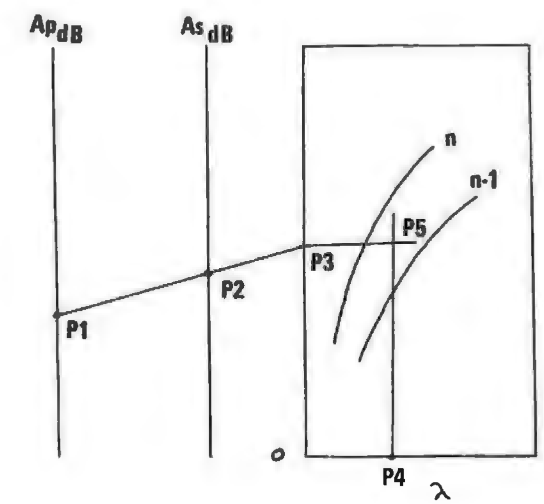


Figure 2-1.3 Nomograph use.

P1 and P2 are the required passband and stopband attenuation, P3 is a turning point, P4 represents the ratio between the frequencies where the stopband attenuation and the passband attenuation are specified, and P5 represents the required filter order, n.

Since n must be an integer, and P5 will generally lie between two integral numbers, always choose the larger of the two integers. Furthermore, if any of the narrowband approximations to bandpass filters are going to be generated, and Chebyshev response is specified, n must be an odd integer. This requirement occurs as even ordered Chebyshev filters have unequal termination resistances, and the narrowband bandpass approximations require equal termination resistances.

These nomographs are also contained in Zverev, however, the two vertical scales appear to be misregistered slightly, and in some applications will give inaccurate results.

These nomographs may also be used in other ways. If the filter order is known, then the filter response may be predicted. In this case, P5 would lie directly on one of the filter order lines, λ and P1, or P2 are the input variables, with P2, or P1 being the output quantity.

Example 2-1.1 Highpass filter

A Butterworth filter is to pass 20 kHz and higher with 3 dB or less attenuation, and reject 10 kHz and lower with at least 40 dB of attenuation. Find the minimum filter order to meet these specifications.

	GSBa	select Butterworth
3.00	GSBA	load Ap_{dB}
40.00	GSBB	load As_{dB}
2.00	GSBC	load λ = (20 kHz)/(10 kHz) = 2, & calculate n
6.65	***	filter order, n (use n = 7)

Example 2-1.2 Bandpass filter

A Chebyshev bandpass filter is centered at 100 kHz (center frequency is not a parameter of the filter order calculation). Frequencies in a 20 kHz passband (geometrically centered about the center frequency) must be passed with 0.5 dB attenuation or less, and frequencies outside a 40 kHz bandwidth (again geometrically centered) must be rejected with at least 40 dB attenuation. Find the minimum filter order to meet these requirements.

	GSBb	select Chebyshev
.50	GSBA	load passband ripple in dB (Ap_{dB})
40.00	GSBB	load minimum stopband attenuation in dB (As_{dB})
2.00	GSBC	load λ = (40 kHz)/(20 kHz) = 2, and calculate n
4.62	***	filter order, n (use n = 5 as smallest integer value to meet specs)

Example 2-1.3 Bandstop example

A maximally flat (Butterworth) bandstop filter is centered at 20 kHz. Frequencies lying outside a 10 kHz band geometrically centered on the center frequency should be attenuated by 3 dB or less. Frequencies inside a band of 1 kHz geometrically centered on the center frequency should be attenuated by at least 60 dB. Find the minimum filter order meeting these specifications.

	GSBa	select Butterworth
3.00	GSBA	load Ap_{dB}
60.00	GSBB	load As_{dB}
10.00	GSBC	load λ = (10 kHz)/(1 kHz) = 10 & calculate n
3.00	***	filter order required
	DSP4	display 4 figures past the decimal point
3.0010	***	the filter order required is greater than 3
3.0000	GSBC	enter filter order of exactly three & calc. λ
10.0079	***	λ where filter is 60 dB down
10.0000	GSBE	enter λ and calculate As_{dB}
59.9794	***	As_{dB} at λ = 10

This bandstop example shows other features of this program. Given n, the ratio, λ , where A_s is met is calculated, and alternately, given λ, A_s for this ratio is calculated.

As an aside, Butterworth filters are not exactly three dB down at the bandedge, but are $10 \cdot \log_{10} 2 = 3.010299957$ dB. If this number had been entered for A_p, the calculated filter order would have been three (to seven significant figures).

Example 2-1.4 Lowpass filter

Find the frequency where a 2 dB ripple, 7th order Chebyshev lowpass filter will be 60 dB down when the cutoff (-2dB) frequency is 1000 Hz.

	GSBb	select Chebyshev
2.00	GSBA	load Ap_{dB}, the passband ripple
60.00	GSBB	load As_{dB}, the minimum stopband rejection
7.00	GSBC	load the filter order, n, and calculate λ
1.78	***	λ to meet above requirements
1000.00	×	cutoff frequency of filter times λ
1781.27	***	frequency where the filter is 60 dB down

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD Ap_{dB}
002	GSB0	
003	STO1	store $A_p^2 - 1$
004	RTN	
005	*LBLB	LOAD As_{dB}
006	GSB0	
007	STO2	store $A_s^2 - 1$
008	RTN	
009	*LBL0	subroutine to convert dB to $(\text{magnitude})^2 - 1$
010	EEX	
011	1	
012	÷	
013	10^x	
014	EEX	
015	-	
016	RTN	
017	*LBLC	LOAD n, THE FILTER ORDER AND CALCULATE λ
018	STO3	
019	RCL2	
020	RCL1	calculate $k = \sqrt{\frac{A_s^2 - 1}{A_p^2 - 1}}$
021	÷	
022	√X	
023	F1?	jump if Butterworth
024	GTO3	
025	GSB2	calculate cosh k
026	RCL3	
027	÷	
028	GSB1	calc $\lambda = \cosh^{-1}(\frac{1}{n}\cosh k)$
029	STO4	
030	GTO8	goto the print/stop routine
031	*LBL3	calculate λ for Butterworth
032	RCL3	
033	1/X	
034	Y^x	$\lambda = (k)^{\frac{1}{n}}$
035	STO4	
036	GTO8	
037	*LBLD	LOAD λ AND CALCULATE n
038	STO4	
039	RCL2	
040	RCL1	calculate k
041	÷	
042	√X	
043	F1?	jump if Butterworth
044	GTO3	
045	GSB2	
046	RCL4	for Chebyshev:
047	GSB2	
048	÷	$n = \frac{\cosh^{-1} k}{\cosh^{-1} \lambda}$
049	STO3	
050	GTO8	
051	*LBL3	
052	LN	for Butterworth:
053	RCL4	
054	LN	$n = \frac{\ln k}{\ln \lambda}$
055	÷	
056	STO3	
057	GTO8	
058	*LBLE	LOAD λ AND CALCULATE As_{dB}
059	F1?	jump if Butterworth
060	GTO3	
061	GSB2	
062	RCL3	for Chebyshev:
063	x	
064	GSB1	$q = \cosh(n \cdot \cosh^{-1} \lambda)$
065	GTO4	
066	*LBL3	for Butterworth:
067	RCL3	$q = (\lambda)^n$
068	Y^x	
069	*LBL4	common part for Buttr & Cheb
070	X^2	
071	RCL1	
072	x	$As_{dB} = 10 \cdot \log((A_p^2 - 1)q^2 + 1)$
073	EEX	
074	+	
075	LOG	
076	EEX	
077	1	
078	x	
079	GTO8	

REGISTERS

0	1	2	3	4	5	6	7	8	9
	$A_p^2 - 1$	$A_s^2 - 1$	n	λ					

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I

Program Listing II

Step	Key	Comment
080	*LBL1	cosh x subroutine
081	e^x	
082	ENT↑	
083	1/X	$\cosh x = \frac{e^x + e^{-x}}{2}$
084	+	
085	2	
086	÷	
087	RTN	
088	*LBL2	$\cosh^{-1} x$ subroutine
089	ENT↑	
090	X^2	
091	EEX	
092	-	
093	√X	$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$
094	+	
095	LN	
096	RTN	
097	*LBLa	SELECT BUTTERWORTH
098	SF1	
099	RTN	
100	*LBLb	SELECT CHEBYSHEV
101	CF1	
102	RTN	
103	*LBLc	SELECT PRINT OR R/S
104	F0?	jump if flag 0 is set
105	GTO3	
106	SF0	
107	1	set flag 0 to indicate print
108	RTN	
109	*LBL3	
110	CF0	clear flag 1 to indicate R/S
111	0	
112	RTN	
113	*LBLd	BANDPASS: enter BW↑ f_o and calculate f_{upr} and f_{lwr}
114	X^2	
115	STO5	
116	X⇄Y	
117	2	
118	÷	
119	ENT↑	
120	X^2	$f_{upr} = \left(\frac{BW}{2}\right) + \sqrt{\left(\frac{BW}{2}\right)^2 + f_o^2}$
121	RCL5	
122	+	
123	√X	
124	+	
125	ENT↑	$f_{lwr} = (f_o)^2/f_{upr}$
126	GSB9	
127	RCL5	
128	X⇄Y	$f_{lwr} = (f_o)^2/f_{upr}$
129	÷	
130	GTO8	
131	*LBLe	BANDPASS: enter f_{upr} & f_{lwr} calculate f_o and BW
132	STO5	
133	X⇄Y	
134	STO6	
135	x	$f_o = ((f_{upr})(f_{lwr}))^{\frac{1}{2}}$
136	√X	
137	GSB9	
138	RCL6	
139	RCL5	$BW = f_{upr} - f_{lwr}$
140	-	
141	*LBL8	print or R/S subroutine
142	GSB9	
143	F0?	if flag 0, space
144	SPC	
145	RTN	
146	*LBL9	
147	F0?	if flag 0, go to print
148	GTO9	
149	R/S	flag 0 not set, R/S
150	RTN	
151	*LBL9	
152	PRTX	
153	RTN	

LABELS

A	B	C	D	E
A_p dB	A_s dB	n→λ	λ→n	λ→A_s dB
a: set Buttr	b: set Chebyshev	c: print, no-print	d: f_o↑BW→fu,fl	e:
0: dB→| |²-1	1: cosh x	2: $\cosh^{-1}$ x	3:	4:
5:	6:	7:	8: print & space	9: print

FLAGS

0	print?
1	Buttr
2	
3	

SET STATUS

FLAGS	TRIG	DISP
--USERS CHOICE--		
0	DEG	FIX
1	GRAD	SCI
2	RAD	ENG
3		n____

PROGRAM 2-2 BUTTERWORTH AND CHEBYSHEV FILTER FREQUENCY RESPONSE AND GROUP DELAY.

Program Description and Equations Used

This program calculates the frequency response (magnitude in dB and phase in degrees) and the un-normalized group delay in seconds for the Butterworth or Chebyshev all pole filter approximations. The response may be in lowpass, highpass, bandpass, or bandstop form (the lowpass and highpass responses are special cases of the bandpass and bandstop responses respectively in that the center frequency is zero). Both single frequency analysis and frequency sweeps may be done. The sweep can be linear using an additive increment, or logarithmic using a multiplicative increment.

The actual analysis routine that is buried within the program analyzes a normalized lowpass filter. The input data is normalized and transformed as required to place it in normalized lowpass form. The phase and gain response (frequency response) of the normalized lowpass filter is the same as the original filter type before transformation; hence, no reverse transformation is necessary for output. The group delay is the rate of change of phase with respect to frequency (derivative of the phase function) and is affected by the transformation to normalized lowpass form, therefore, an output transformation from the normalized lowpass group delay is required.

The logarithm of the normalized lowpass filter transmission function, $T(j\Omega)$ is composed of two components, the attenuation, a, and the phase, b. As a complex number, these two components represent the constant, g:

$$T(j\Omega) = \prod_{k} \frac{K}{\sigma_k + j\,(\omega_k - \Omega)} \qquad (2.2.1)$$

$$g = \ln(T(j\Omega)) = a + jb \qquad (2\text{-}2.2)$$

$$\Omega = F(\omega) \qquad (2\text{-}2.3)$$

$$\omega = 2\pi f \qquad (2\text{-}2.4)$$

where $F(\omega)$ represents the transformation to normalized lowpass, and σ_k and ω_k are the pole locations of the Butterworth or Chebyshev normalized lowpass transfer function (see the equation derivation section following the examples for pole location details).

The group delay of the normalized lowpass filter is the derivative of the phase function, b, taken with respect to radian frequency:

$$b = \sum_{k=1}^{n} \tan^{-1}\left\{\frac{\omega_k - \Omega}{\sigma_k}\right\} \qquad (2\text{-}2.5)$$

$$\tau_{g_{nor}} = \frac{db}{d\Omega} = \sum_{k=1}^{n} \frac{|\sigma_k|}{\sigma_k^2 + (\omega_k - \Omega)^2} \qquad (2\text{-}2.6)$$

The group delay is denormalized by multiplying the normalized group delay, Eq. (2-2.6), by the derivative of the transformation function, Eq. (2-2.3), taken with respect to the un-normalized radian frequency, ω.

$$\tau_g = \tau_{g_{nor}} \cdot \frac{d\Omega}{d\omega} \qquad (2\text{-}2.7)$$

The transform functions for the bandpass and lowpass cases are:

$$\Omega_{BP} = \left|\frac{1}{BW}\left\{f - \frac{f_o^2}{f}\right\}\right| \qquad (2\text{-}2.8)$$

$$\frac{d\Omega_{BP}}{d\omega} = \frac{1}{2\pi BW}\left\{1 + \frac{f_o^2}{f^2}\right\} \qquad (2\text{-}2.9)$$

where "BW" and "f_o" are the bandwidth and center frequency of the bandpass filter in hertz, and "f" is the frequency to be transformed (in hertz). The center frequency is zero for the lowpass case.

The transform functions for the bandstop and highpass cases are:

$$\Omega_{BS} = \frac{1}{\Omega_{BP}} = \left|\frac{BW}{f - \frac{f_o^2}{f}}\right| \qquad (2.2.10)$$

$$\frac{d\Omega_{BS}}{d\omega} = \frac{BW}{2\pi}\left\{\frac{f^2 + f_o^2}{(f^2 - f_o^2)^2}\right\} \qquad (2\text{-}2.11)$$

The definitions of the terms are the same as above, and the highpass case has zero center frequency also.

The program uses Eqs. (2-2.8) and (2-2.10) to transform the input data to normalized lowpass, and then evaluates Eqs. (2-2.1) and (2-2.6) to obtain the frequency response and normalized lowpass group delay. The group delay is denormalized using Eqs. (2-2.9) or (2-2.11), and the frequency response and group delay are printed (HP-97 only) and displayed.

2-2 # User Instructions

BUTTERWORTH AND CHEBYSHEV FILTER GROUP DELAY				
C: n↑DBR	HP: f_o	BP: BW↑f_o	lin: 0 log: 1	f_{start}↑Δf
B: n	LP: f_o	BS: BW↑f_o	start sweep	single freq analysis

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	a) if Butterworth, enter filter order	n	[A]	
	b) if Chebyshev:			
	enter passband ripple in dB	DBR	[ENT↑]	
	enter filter order	n	[f] [A]	
3	Select filter type and enter characteristics			
	a) if lowpass: enter cutoff frequency*	BW, Hz	[B]	
	b) if highpass: enter cutoff frequency*	BW, Hz	[f] [B]	
	c) if bandstop:			
	enter bandwidth**	BW, Hz	[ENT↑]	
	enter center frequency	f_o, Hz	[C]	
	d) if bandpass:			
	enter bandwidth**	BW, Hz	[ENT↑]	
	enter center frequency	f_o, Hz	[f] [C]	
4	If sweep of frequencies is desired:			
	a) select linear or logarithmic sweep (toggle)		[f] [D]	0
			[f] [D]	1
			[f] [D]	0
				⋮
	b) enter sweep starting frequency in hertz	f_{start}	[ENT↑]	
	c) enter frequency increment:	f	[f] [E]	
	if linear sweep, the increment is			
	additive; if logarithmic, the			
	increment is multiplicative.			
	d) start sweep		[D]	f, Hz
				loss, dB
				phase, deg
				group delay sec
5	for analysis at a single frequency	f, Hz	[E]	analysis above
	NOTE (* & **)			
	The LP & HP cutoff frequency and the			
	BP & BS bandwidth are defined as the			
	-3dB point for Butterworth, and the -DBR			
	point for Chebyshev			

Example 2-2.1

Calculate the amplitude, phase, and group delay characteristics of a third order, 1 dB ripple Chebyshev bandpass filter with 1000 Hz bandwidth and 10000 Hz center frequency. Calculate these characteristics from 8000 Hz to 12000 Hz in linear increments of 100 Hz.

PROGRAM INPUT	PROGRAM OUTPUT				
	8.000+03	9.000+03	10.00+03	11.00+03	12.00+03
	-45.04+00	-24.06+00	0.000+00	-21.06+00	-39.53+00
	-257.1+00	-240.1+00	0.000+00	-235.9+00	-254.0+00
3. ENT↑ n	21.23-06	110.6-06	802.3-06	121.1-06	21.89-06
1. GSBa DBR					
	8.100+03	9.100+03	10.10+03	11.10+03	frequency
1000. ENT↑ BW	-43.48+00	-20.73+00	-345.5-03	-23.78+00	20 log|H(jω)|
10000. GSBc fo	-256.3+00	-235.4+00	-27.82+00	-239.7+00	∡ H(jω), deg
↲ bandpass	23.69-06	151.2-06	723.6-06	92.14-06	τ_g, sec
GSBd } linear					
0.000+00 *** } sweep	8.200+03	9.200+03	10.20+03	11.20+03	
	-41.85+00	-16.90+00	-894.2-03	-26.20+00	
8000. ENT↑ fstart	-255.4+00	-228.8+00	-51.87+00	-242.6+00	
100. GSBe Δf	26.62-06	223.1-06	624.7-06	72.90-06	
GSBD start	8.300+03	9.300+03	10.30+03	11.30+03	
analysis	-40.13+00	-12.35+00	-906.8-03	-28.38+00	
	-254.4+00	-218.5+00	-74.49+00	-245.0+00	
	30.17-06	371.0-06	667.2-06	59.38-06	
	8.400+03	9.400+03	10.40+03	11.40+03	
	-38.31+00	-6.901+00	-195.5-03	-30.36+00	
	-253.3+00	-199.6+00	-103.5+00	-247.0+00	
	34.53-06	731.4-06	1.006-03	49.47-06	
	8.500+03	9.500+03	10.50+03	11.50+03	
	-36.37+00	-1.466+00	-654.3-03	-32.17+00	
	-251.9+00	-160.9+00	-148.3+00	-248.6+00	
	39.96-06	1.430-03	1.371-03	41.94-06	
	8.600+03	9.600+03	10.60+03	11.60+03	
	-34.30+00	-82.00-03	-4.904+00	-33.85+00	
	-250.4+00	-109.8+00	-189.4+00	-250.0+00	
	46.88-06	1.189-03	844.2-06	36.08-06	
	8.700+03	9.700+03	10.70+03	11.70+03	
	-32.07+00	-865.9-03	-9.967+00	-35.42+00	
	-248.5+00	-76.75+00	-211.4+00	-251.2+00	
	55.92-06	726.3-06	432.4-06	31.41-06	
	8.800+03	9.800+03	10.80+03	11.80+03	
	-29.66+00	-909.1-03	-14.30+00	-36.87+00	
	-246.3+00	-52.79+00	-223.3+00	-252.3+00	
	68.11-06	648.4-06	253.9-06	27.62-06	
	8.900+03	9.900+03	10.90+03	11.90+03	
	-27.01+00	-351.4-03	-17.94+00	-38.24+00	
	-243.6+00	-28.08+00	-230.8+00	-253.2+00	
	85.24-06	737.1-06	168.4-06	24.50-06	

Equations Used and Pole Locations

Butterworth pole locations: The pole locations of a normalized lowpass Butterworth filter lie on a circle in the complex plane. Odd ordered filters have a real pole plus complex conjugate pairs. Even order filters have only complex conjugate pairs. No poles ever lie directly on the $j\omega$ axis. Figure 2-2.1 shows the pole locations for a 5th order normalized Butterworth lowpass filter, and Eqs. (2-2.12) and (2-2.13) show the generalized pole locations.

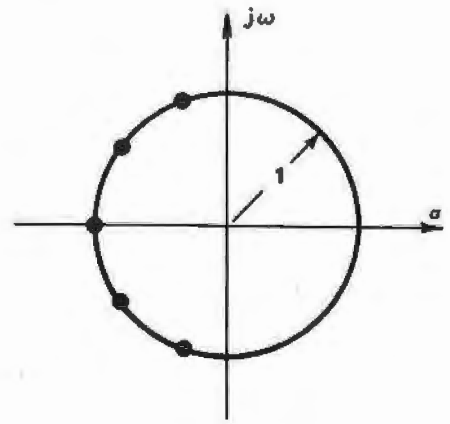


Figure 2-2.1 Butterworth pole locations.

Pole locations:

$$\text{Real part,}\quad \sigma_k = -\sin\left(\frac{2k-1}{2n}\pi\right) \qquad (2\text{-}2.12)$$

$$\text{Imag part,}\quad \omega_k = \cos\left(\frac{2k-1}{2n}\pi\right) \qquad (2\text{-}2.13)$$

$k = 1, 2, \ldots, n$

(trig argument is in radians)

The attenuation of the normalized Butterworth lowpass filter is 3 dB at $\omega = 1$. At other frequencies, the attenuation in dB is expressed by:

$$A_{dB} = 10\log\left(1 + \omega^{2n}\right) \qquad (2\text{-}2.14)$$

As shown by this equation, the attenuation monotonically increases as frequency increases. Figure 2-2.2 shows the general shape of the Butterworth response.

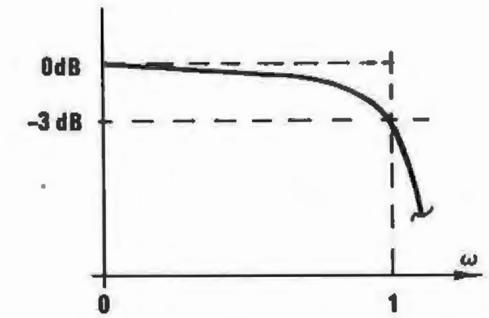


Figure 2-2.2 Normalized Butterworth amplitude response.

Chebyshev pole locations: The normalized lowpass pole locations of a Chebyshev lowpass filter lie on an ellipse with major axis dimension cosh a, and minor axis dimension sinh a where a is defined by:

$$a = 1/n\ \sinh^{-1}(1/\varepsilon) \qquad (2\text{-}2.15)$$

The parameter ε is related to the passband ripple in dB by:

$$\varepsilon = \left(10^{0.1\varepsilon_{dB}} - 1\right)^{\frac{1}{2}} \qquad (2\text{-}2.16)$$

Using these quantities, the real and imaginary parts of the pole locations are given by Eqs. (2-2.17) and (2-2.18). Figure 2-2.3 shows the pole locations for a fifth order Chebyshev filter.

$$\text{Real part,}\quad \sigma_k = -(\sinh a)\left(\sin\frac{2k-1}{2n}\pi\right) \qquad (2\text{-}2.17)$$

$$\text{Imag part,}\quad \omega_k = (\cosh a)\left(\cos\frac{2k-1}{2n}\pi\right) \qquad (2\text{-}2.18)$$

$k = 1, 2, \ldots, n$

(trig argument is in radians)

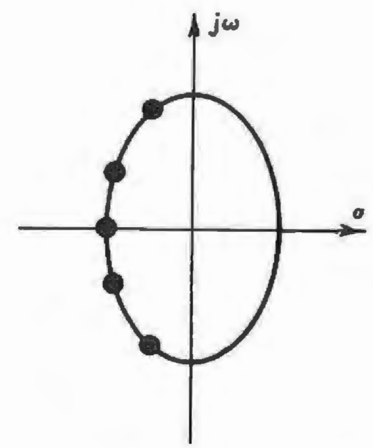


Figure 2-2.3 Chebyshev pole locations (5th order).

The passband edge of a Chebyshev filter is defined as the highest frequency where the response is ε_{dB} down. Remember, the Chebyshev passband response oscillates within a band of ε_{dB}. Fourth and fifth order Chebyshev responses are shown in Fig. 2-2.4.

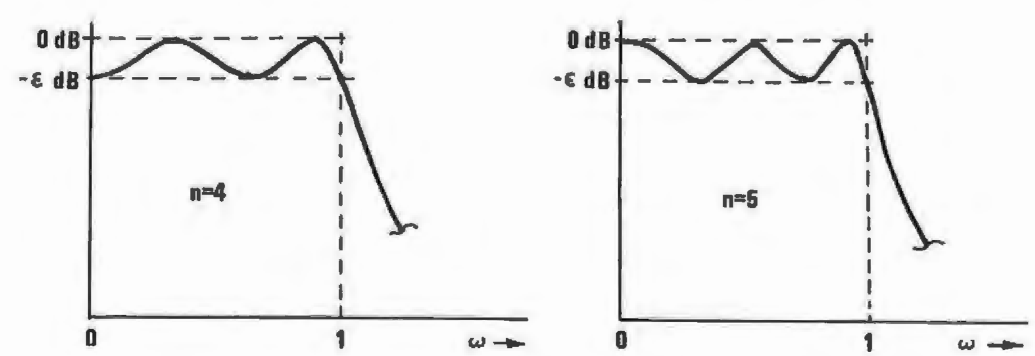


Figure 2-2.4 Chebyshev normalized lowpass filter responses.

The normalized frequency where the Chebyshev filter response is 3 dB down is given by the expression:

$$f_{-3dB} = \cosh\left\{ \frac{1}{n} \cosh^{-1}\left(\frac{1}{\varepsilon}\right) \right\} \qquad (2\text{-}2.19)$$

Comparing the equations that define the pole locations for the Butterworth and Chebyshev filters, one will notice that the only difference is the Chebyshev poles are modified by hyperbolic functions. If the sinh a and cosh a functions are defined to be unity, then the

Chebyshev equations become the Butterworth equations. This technique is used in the program. Chebyshev poles are always calculated; however, if Butterworth response is selected, the hyperbolic functions are not calculated, but are set equal to one in register storage.

Another difference between Butterworth and Chebyshev filters lies in the definition of the bandedge. Butterworth response is 3 dB down at the bandedge, and Chebyshev response is ε_{dB} down at the bandedge where ε_{dB} is the passband ripple in dB. Flag 1 is used to indicate the filter type, and is set for Butterworth. When the pole locations are calculated, flag 1 is tested to see what equation, if any, is to be used to convert the given passband edge frequency into the appropriate frequency for the filter type being used.

Program Listing I

```
001  *LBLA  LOAD BUTTERWORTH FILTER ORDER
002  STO1   store n
003  EEX
004  STO5   cosh = 1 for Butterworth
005  STO6   sinh = 1   "       "
006  RCL1   recall n to display
007  RTN
008  *LBLa  LOAD CHEB ORDER AND DB RIPPLE
009  STOC   store dB ripple
010  R↓
011  STO1   store n
012  RCLC   calculate epsilon, ε
013  EEX        ε = √(10^(0.1 Amax) - 1)
014  1
015  ÷
016  10^x
017  EEX
018  -
019  √X
020  1/X    calculate sinh^-1(1/ε)
021  ENT↑
022  X²
023  EEX
024  +
025  √X
026  +
027  RCL1   calculate sinh((1/n) sinh^-1(1/ε))
028  1/X
029  Y^x        sinh^-1 x = ln(x + √(x²+1))
030  ENT↑
031  ENT↑       Y^(1/n x) ≡ e^((1/n) ln x)
032  ENT↑
033  1/X        sinh x = (e^x - e^-x)/2
034  -
035  STO6
036  R↓     calculate cosh((1/n) sinh^-1(1/ε))
037  1/X
038  +          cosh x = (e^x + e^-x)/2
039  STO5
040  2
041  ST÷5
042  ST÷6
043  RTN
044  *LBLB  LOAD fo FOR LOWPASS CASE
045  CF0
046  GTO1
047  *LBLb  LOAD fo FOR HIGHPASS CASE
048  SF0
049  *LBL1
050  STO3   store fo
051  CLX        fo² ≡ 0 for lowpass and
052  STO2       highpass cases
053  RCL3
054  RTN
055  *LBLC  LOAD BW AND fo FOR BANDSTOP
056  SF0
057  GTO1
058  *LBLc  LOAD BW AND fo FOR BANDPASS
059  CF0
060  *LBL1
061  X²     calc & store fo²
062  STO2
063  R↓
064  STO3   store bandwidth
065  RTN
066  *LBLD  START SWEEP
067  SPC
068  *LBL7
069  RCL8
070  PRTX
071  GSBE
072  RCL9
073  F1?
074  GTO1
075  ST+8   linear sweep increment
076  GTO7
077  *LBL1
078  ST×8   log sweep increment
079  GTO7
080  *LBLd  SELECT LIN/LOG SWEEP
081  F1?
082  GTO1
083  SF1
084  EEX
085  RTN
086  *LBL1
087  CF1
088  CLX
089  RTN
090  *LBLe  LOAD SWEEP f_start AND Δf
091  STO9
092  R↓
093  STO8
094  RTN
```

REGISTERS

0 present frequency	1 n	2 f_o^2	3 bandwidth	4 f	5 cosh	6 sinh	7 Σ delay	8 $\Pi \frac{\sigma^2+\omega^2}{\sigma^2+(\omega-\Omega)^2}$	9 Σ phase
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A σ_k	B ω_k	C A_{max}	D Δf	E Ω	I index

Program Listing II

```
095  *LBLE  LOAD ANALYSIS FREQUENCY
096  STO4   store frequency and form:
097  RCL2
098  RCL4       Ω = (1/BW) {f - fo²/f}
099  ÷
100  -
101  RCL3
102  ÷
103  F0?    if bandstop, 1/Ω → Ω
104  1/X
105  ABS    store |Ω|
106  STOE
107  RCL1   initialize loop:
108  STOI       n → RI
109  CLX        Σ7 = 0
110  STO7       Π8 = 1
111  STO9       Σ9 = 0
112  EEX
113  STO8
114  *LBL0
115  RCLI
116  ENT↑
117  +
118  EEX    calculate angle:
119  -
120  RCLI       θk = 90((2k-1)/n)
121  ÷
122  9
123  0
124  ×
125  EEX    calculate sin θk & cos θk
126  →R
127  RCL5   form and store ωk
128  ×
129  STOB
130  RCLE   form: ωk - Ω
131  -
132  X⇄Y
133  RCL6   form and store σk
134  ×
135  STOA
136  →P     form and sum:
137  X²         σk / (σk² + (ωk - Ω)²)
138  RCLA
139  X⇄Y
140  ÷
141  ST+7
142  RCLA
143  ÷      form: 1 / (σk² + (ωk - Ω)²)
144  ST×8
145  X⇄Y
146  ST+9   sum phase element
147  RCLA   form in R8:
148  X²
149  RCLB       Π(k=1..n) {(σk² + ωk²) / (σk² + (ωk - Ω)²)}
150  X²
151  +
152  ST×8
153  DSZI   decrement k and test
154  GTO0   for loop exit
155  RCL7
156  Pi     calculate: Σ7 / 2π
157  ENT↑
158  +
159  ÷
160  RCL3
161  F0?    jump if highpass
162  GTO8   or bandstop
163  ÷
164  RCL2
165  RCL4   lowpass or bandpass dΩ/dω
166  X²
167  ÷          τg = {1 + fo²/f²} Σ7/(2π BW)
168  EEX
169  +
170  ×
171  GTO9
172  *LBL8
173  ×
174  RCL4
175  X²     highpass or bandstop dΩ/dω
176  RCL2
177  +          τg = {(f² + fo²)/(f² - fo²)²} · (BW/2π) · Σ7
178  ×
179  RCL4
180  X²
181  RCL2
182  -
183  X²
184  ÷
185  *LBL9
186  RCL8
187  LOG    calculate and print
188  EEX    amplitude response
189  1      in dB
190  ×
191  PRTX
192  R↓
193  RCL9   calculate and print
194  F0?    phase response in degrees
195  CHS
196  PRTX
197  R↓
198  PRTX   print group delay
199  SPC
200  RTN
```

LABELS

A BUTTERWORTH n	B LP: f_o	C BS: BW↑f_o	D START SWEEP	E $f \to \tau_g$, etc
a CHEBYSHEV n↑dB Ripple	b HP: f_o	c BP: BW↑f_o	d SELECT LOG/LIN SWP	e f_{START}↑Δf
0 SUMMATION LOOP START	1 MULTIUSE LABEL	2	3	4
5	6	7 SWEEP START	8 BS, HP OUTPUT	9 PRINT & SPACE SUBROUTINE

FLAGS

0	CLR: LP or BP; SET: HP or BS
1	CLR: LINEAR; SET: LOG
2	
3	

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0		■	DEG	■	FIX	
1		■	GRAD		SCI	
2			RAD		ENG	■
3					n 3	

Suggested HP-67 program changes. The "print" command is used to output data in the program listing. These print commands are located at the following line numbers: 070, 191, 196, and 198. HP-67 users may prefer either a "pause" or "R/S" command replacing the "print" command at the above line numbers. If the R/S change is made, the program execution will stop at each data output point. To resume program execution, execute a "R/S" command from the keyboard.

PROGRAM 2-3 BUTTERWORTH AND CHEBYSHEV LOWPASS NORMALIZED COEFFICIENTS.

Program Description and Equations Used

This program calculates the normalized (1 ohm, 1 radian/second cutoff) element values for either the Butterworth (maximally flat) or Chebyshev (equal ripple passband) all pole lowpass filter approximations. The filters can be either doubly terminated (resistors at both ends) or singly terminated (driven from a voltage or current source, i.e., R_T approaches infinity). Because of duality, two filter topologies exist for the ladder filter as shown in Fig. 2-3.1. These topologies are bilateral and passive; therefore, the voltage source can be placed in series with the left-hand termination resistor as shown, or in series with the right-hand termination resistor. By proper selection of the filter topology and input port designation, the singly terminated filter can be driven from either a current or voltage source and resistively terminated, or driven from a Thevenin (or Norton) equivalent source and terminated in either a short or open circuit.

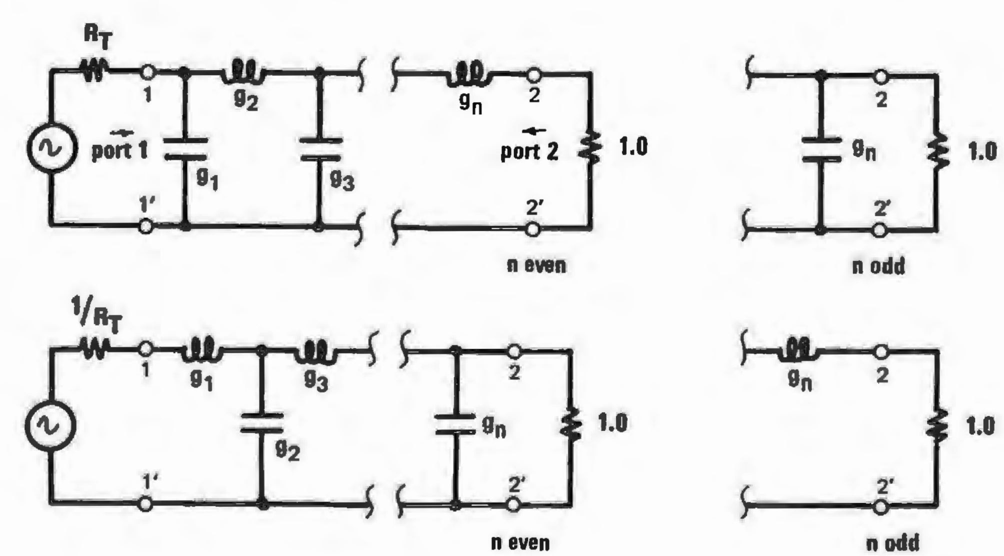


Figure 2-3.1 Lowpass ladder filter topologies.

The search for explicit formulas for ladder filter element values has extended over four decades. Bennett [7] provided the remarkedly simple formula for equally terminated Butterworth filters in 1932. Norton [39] provided the formulas for the open circuited Butterworth case in 1937. Belevitch [5] published formulas for the doubly terminated Chebyshev case in 1952. Orchard [40] gathered together this previous work and provided the missing fourth formula set for the open circuited Chebyshev case in 1953. Green [28] went on to generalize these formulas for any ratio of resistive terminations in 1954. These formulas had been numerically tested, but never formally proved. Doyle [22] provided a "hammer and tongs" brute force proof for the Butterworth case with arbitrary terminations. Meanwhile, in Japan, Takahasi [51], had made an ingenious proof of the formulas for the arbitrarily terminated Chebyshev case and extended it to the Butterworth case by a limiting process. Takahasi published his independent work in 1951 (in Japanese), but it was not discovered by the rest of the world until 1957. Weinberg and Slepian [54] discuss Takahasi's results. Takahasi's results can also be found in the back of Weinberg's book [53].

The recursion relations given by Eqs. (2-3.1) through (2-3.16) are adapted from Takahasi. If the filter order is odd, the filter can be terminated by 1 ohm at one port and by any resistance 1 ohm or larger at the other port. By using the dual topology, the termination resistance can be any resistance 1 ohm or smaller (including 0 ohms). If the filter is an even ordered Chebyshev design, then the first port termination resistance must be larger than 1 ohm. The minimum value of this termination resistance is given by Eq. (2-3.18).

Takahasi's recursion relationships:

$$g_{r+1} = \frac{A \cdot s_{r-\frac{1}{2}} \cdot s_{r+\frac{1}{2}}}{g_r\,(\xi^2 + \eta^2 - \xi\eta\, c_r + s_r^2)} \qquad (2\text{-}3.1)$$

where $r = 1, 2, \ldots, n-1$

$$g_1 = \frac{\sqrt{A} \cdot s_{\frac{1}{2}}}{R_T\,(\xi - \eta)} \qquad (2\text{-}3.2)$$

$$s_q = 2 \cdot \sin\left(\frac{\pi \cdot q}{n}\right) \qquad (2\text{-}3.3)$$

$$c_q = 2 \cdot \cos\left(\frac{\pi \cdot q}{n}\right) \qquad (2\text{-}3.4)$$

For normalized lowpass Butterworth coefficients:

$$A = 1 \qquad (2\text{-}3.5)$$

$$\xi = 1 \qquad (2\text{-}3.6)$$

$$\eta = \left(\frac{R_T - 1}{R_T + 1}\right)^{1/n} \qquad (2\text{-}3.7)$$

$$s_r^2 \equiv 0 \qquad (2\text{-}3.8)$$

For normalized lowpass Chebyshev coefficients:

$$A = 4 \qquad (2\text{-}3.9)$$

$$\xi = F(1) \qquad (2\text{-}3.10)$$

$$\eta = F\left(1 - \frac{4 \cdot v\, R_T}{(1 + R_T)^2}\right) \qquad (2\text{-}3.11)$$

$$v = \begin{cases} 1 + \xi^2, & n \text{ even} \\ 1, & n \text{ odd} \end{cases} \qquad (2\text{-}3.12)$$

$$F(x) = u - \frac{1}{u} \qquad (2\text{-}3.13)$$

$$u = \left(\sqrt{\frac{x}{\varepsilon^2}} + \sqrt{\frac{x}{\varepsilon^2} + 1}\right)^{1/n} \qquad (2\text{-}3.14)$$

$$y = 10^{\varepsilon_{dB}/20} \qquad (2\text{-}3.15)$$

$$\varepsilon^2 = y^2 - 1 \qquad (2\text{-}3.16)$$

$$\omega_{-3dB} = \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right) \qquad (2\text{-}3.17)$$

$$R_L\Big|_{\substack{min \\ n\ even}} = \left(\frac{\sqrt{\frac{y+1}{y-1}} - 1}{\sqrt{\frac{y+1}{y-1}} + 1}\right)^2 \qquad (2\text{-}3.18)$$

When the termination ratio is neither 0, ∞ , or as close as possible to 1, there are more than one possible set of ladder element values for the same filtering function. These alternate sets are synthesized by realizing the reflection zeros in the RHP, or RHP-LHP alternating rather than in the LHP. The closed form formulas realize the LHP reflection zero case. This realization generally results in a ladder filter with minimum sensitivity to component value changes. For a more comprehensive discussion of reflection zeros and order of realization, see Weinberg [53], chapter 13.

The program is set up to calculate the minimum termination resistance, and if the value loaded by the user is less than the minimum, the minimum value replaces the user loaded value.

When the termination resistance is allowed to approach infinity (or 0 using the dual topology), the filter only has one termination resistor, and is called "singly terminated." These singly terminated filters are used where it is inconvenient, or wasteful of power, to use the doubly terminated filter. Because the loaded Q's of the resonant circuits become higher as the unloaded end of the filter is approached, the singly terminated design is more difficult to align.

Often, the LC filter is used as a basis for an active filter design such as Szentirmai's leapfrog topology [48], Bruton's frequency dependent negative resistor (FDNR) approach [10], or Orchard and Sheahans' type 11 active simulation [42]. Using the doubly terminated LC topologies for the active filter basis, will also mean that the active filters will be less critical toward alignment.

User Instructions

BUTTERWORTH AND CHEBYSHEV LP NORMALIZED COEFFICIENTS				
Load n	Load R_T ($R_T \geq 1$)	calculate Butterworth values	calculate $-\epsilon_{dB}$ Cheb values	calculate -3_{dB} Cheb values

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Load filter order (n= 12 maximum)	n	A	n
	If the normalized lowpass prototype is			
	to be transformed to bandpass types			
	6, 7, 8, 9, 10, or 11, and Chebyshev			
	response is desired, the filter order			
	must be odd so the terminations will be			
	equal resistance.			
3	Load the termination resistance desired	R_T	B	
	The termination resistance must be			
	1 or larger. For terminations less			
	than 1 ohm (normalized) load the			
	reciprocal value and use the dual			
	topology. See note after step 7.			
4	For Butterworth coefficients		C	R_T
				space
				g_1
				g_2
				⋮
				g_n
				space
				$R_L = 1$
5	For Chebyshev coefficients that define a	ϵ_{dB}	D	$\omega_{-3\ dB}$
	filter that is $-\epsilon$ dB down at $\omega = 1$			space
	If even ordered Chebyshev has been			R_T
	selected, the minimum source resistance			space
	is calculated and is used if the			g_1
	resistance loaded in step 3 is smaller.			g_2
				⋮
				g_n
				space
				$R_L = 1$

2-3 User Instructions

BUTTERWORTH AND CHEBYSHEV LP NORMALIZED COEFFICIENTS

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
6	For Chebyshev coefficients that define a	ϵ_{dB}	E	$\omega_{-\epsilon}$ dB
	filter that is 3 dB down at $\omega = 1$			space
	The minimum source resistance comment			R_T
	for even ordered Chebyshev filters			space
	in step 5 also applies here.			g_1
				g_2
				:
				g_n
				space
				R_L 1
7	Go back and repeat any step.			
	The last calculated coefficients will			
	be in storage for use by other programs			
	in this section.			
	Notes on termination resistance:			
	To enable the program to output coefficients for the singly terminated case, load 10^5 ohms for R_T if Chebyshev response is going to be selected, or load 10^9 ohms if Butterworth response is going to be selected. Either one of these values is a reasonable approximation to infinity when compared to one ohm. The maximum termination resistance in the Chebyshev case is limited to 10^5 ohms because of a small difference between big numbers problem. 10^5 ohms is a compromise between an approximation to infinity and the number of significant digits in the coefficients. With 10^5 ohms, the answers are significant to five places.			

Example 2-3.1

Find the normalized lowpass coefficients for a 4th order, ½ dB ripple Chebyshev filter that is doubly terminated, and has the minimum termination resistance. The filter response should be 3 dB down at the passband edge ($\omega = 1$) relative to the response at dc.

HP-97 printout

```
           4. GSBA   load filter order
           1. GSBB   load termination resistance desired
           .5 GSBE   enter passband ripple in dB and calculate
                     Chebyshev coefficients
914.826-03   ***     ω-εdB (output)

1.98406+00   ***     minimum termination resistance allowed at port 1

920.243-03   ***     g1
2.58646+00   ***     g2
1.30355+00   ***     g3
1.62581+00   ***     g4

1.00000+00   ***     port 2 termination resistance
```

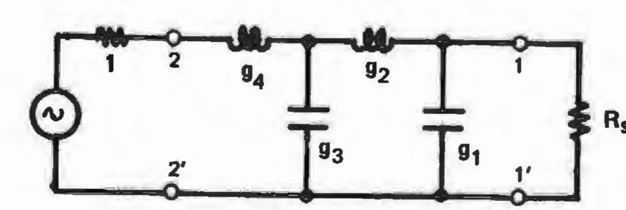


Figure 2-3.2 One topology for normalized lowpass filter (port ordering reversed).

Example 2-3.2

Find the normalized lowpass coefficients for a 10th order Butterworth filter that is singly terminated.

HP-97 printout

10. GSBA		load filter order
1.+05 GSBE		load termination resistance (use 10^5 for Chebyshev)
GSBC		calculate Butterworth coefficients
1.00000+03	***	termination resistance at port 1
1.56434+00	***	g_1
1.85516+00	***	g_2
1.81211+00	***	g_3
1.55263+00	***	g_4
1.51002+00	***	g_5
1.[illegible]203+00	***	g_6
1.04062+00	***	g_7
762.527-03	***	g_8
465.375-03	***	g_9
156.434-03	***	g_{10}
1.00000+00	***	termination resistance at port 2

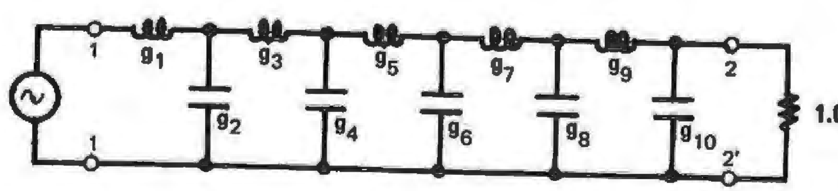


Figure 2-3.3 One form for normalized lowpass filter (dual topology used).

2-3

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD FILTER ORDER
002	STO6	store filter order
003	Pi	
004	X⇄Y	calculate and store π/n
005	÷	
006	STO1	
007	RCLE	
008	RTN	
009	*LBLE	LOAD DESIRED TERMINATION RESISTANCE
010	STOD	
011	RTN	
012	*LBLC	CALCULATE BUTTERWORTH COEFS
013	CF1	
014	EEX	set $\omega_{-3dB} = 1$
015	STO0	
016	STO2	set $\xi = 1$
017	RCLD	calculate and store:
018	EEX	
019	-	
020	RCLD	
021	EEX	$\eta = \left\{\frac{R_T - 1}{R_T + 1}\right\}^{\frac{1}{n}} \rightarrow R_3$
022	+	
023	÷	
024	GSB4	
025	STO3	
026	GTO0	jump around Cheb setup
027	*LBLE	CALCULATE -3dB CHEBYSHEV COEFFICIENTS
028	SF0	
029	GTO2	
030	*LBLD	CALCULATE - dB CHEBYSHEV COEFFICIENTS
031	CF0	
032	*LBL2	
033	SF1	indicate Chebyshev
034	EEX	
035	1	calculate and store:
036	÷	
037	10^x	
038	EEX	$\epsilon^2 = 10^{\epsilon_{dB}/10} - 1 \rightarrow R_3$
039	-	
040	STO3	
041	EEX	calculate and store:
042	+	$y = 10^{\epsilon_{dB}/20} \rightarrow R_E$
043	√X	
044	STOE	
045	EEX	calculate:
046	+	
047	RCLE	
048	EEX	
049	-	
050	÷	$R_{T_{min}} = \left[\frac{\sqrt{\frac{y+1}{y-1}} - 1}{\sqrt{\frac{y+1}{y-1}} + 1}\right]^2$
051	√X	
052	STOE	
053	EEX	
054	+	
055	RCLE	
056	EEX	
057	-	
058	÷	
059	X^2	
060	RCLD	if filter order is even, and R_T desired is less than $R_{T_{min}}$, replace R_T desired by $R_{T_{min}}$
061	X≤Y?	
062	R↓	
063	GSB6	
064	F2?	
065	F2?	
066	STOD	
067	EEX	calculate and store:
068	GSB7	$\xi = F(1) \rightarrow R_2$
069	STO2	
070	RCL3	calculate and store:
071	1/X	
072	√X	
073	LSTX	
074	EEX	
075	-	
076	√X	$\omega_{-3dB} = \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\epsilon}\right) \rightarrow R_0$
077	+	
078	GSB4	
079	1/X	
080	+	
081	2	
082	÷	
083	STO0	
084	F0?	calculate $\omega_{-\epsilon dB}$ if ω_{-3dB} coefficients requested
085	1/X	
086	GSB3	print ω_{-3dB}, or $\omega_{-\epsilon dB}$
087	GSB6	calculate
088	RCL3	$v = \begin{cases} 1+\epsilon^2, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$
089	EEX	
090	+	
091	F2?	
092	LSTX	
093	4	calculate and store:
094	x	
095	RCLD	
096	x	
097	RCLD	
098	EEX	
099	+	
100	X^2	$\eta = F\left(1 - \frac{4vR_T}{(1+R_T)^2}\right) \rightarrow R_3$
101	÷	
102	CHS	
103	EEX	
104	+	
105	X<0?	X<0 default
106	CLX	
107	GSB7	
108	STO3	
109	*LBL0	
110	RCLD	
111	P⇄S	
112	CLRG	clear coefficient registers

REGISTERS

0: ω_{-3dB}	1: $\frac{\pi}{n}$	2: ξ, 1	3: $\epsilon^2, \eta, \lambda$	4: g_r	5:	6: n	7: r	8: $\frac{2r-1}{2}$	9: S_{2r-1}
S0: g_1	S1: g_2	S2: g_3	S3: g_4	S4: g_5	S5: g_6	S6: g_7	S7: g_8	S8: g_9	S9: g_{10}
A: g_{11}	B: g_{12}	C: g_{13}	D: R_T	E: $y, \sqrt{\frac{y+1}{y-1}}, c_{2r}$	I: index				

Program Listing II

NOTE TRIG MODE

Step	Key	Comment
113	P⇄S	
114	STOD	
115	GSB3	print actual termination R
116	1/X	
117	STO4	initialize registers
118	EEX	
119	1	
120	STOI	
121	EEX	
122	STO7	
123	.	
124	5	
125	STO8	calculate and store:
126	GSB8	
127	STO9	
128	ENT↑	$g_1 = \frac{A^{1/2} \cdot g_{1/2}}{R_T(\xi - \eta)}$
129	F1?	
130	+	
131	ST×4	if Chebyshev use A = 4,
132	RCL2	otherwise use A = 1
133	RCL3	
134	GSB5	
135	*LBL1	recursion loop start
136	ISZI	increment register index
137	RCL9	$s_{r+1/2}$ start g_{r+1} calculation
138	ST×4	
139	EEX	
140	ST+8	
141	RCL8	
142	GSB8	$s_{r-1/2}$
143	ST×4	
144	STO9	
145	4	
146	F1?	if Chebyshev, use A = 4
147	ST×4	
148	RCL7	finish g_{r+1} calculation
149	GSB8	
150	X²	
151	RCL2	
152	X²	
153	F1?	add s_r^2 if Chebyshev
154	+	
155	RCL3	
156	X²	
157	+	
158	RCLE	
159	RCL2	
160	×	
161	RCL3	
162	×	
163	GSB5	
164	EEX	increment r
165	ST+7	
166	RCL7	
167	RCL6	test for loop exit
168	X>Y?	
169	GTO1	
170	SPC	$R_L = 1$
171	EEX	
172	*LBL3	print and space subroutine
173	PRTX	
174	SPC	
175	RTN	
176	*LBL5	subroutine to finish g_{r+1}
177	-	
178	ST÷4	calculation, store result,
179	RCL0	
180	RCL4	and setup g_r for next
181	ST÷4	
182	ST÷4	iteration
183	F0?	
184	×	
185	STOi	
186	PRTX	
187	RTN	
188	*LBL6	subroutine to set flag 2
189	RCL6	if filter order is odd
190	2	
191	÷	
192	FRC	
193	X≠0?	
194	SF2	
195	R↓	
196	RTN	
197	*LBL7	subroutine to calculate:
198	RCL3	
199	÷	
200	√X	
201	ENT↑	$F(x) = u - \frac{1}{u}$
202	X²	
203	EEX	
204	+	
205	√X	
206	+	$u = \left\{\sqrt{\frac{x}{\epsilon^2}} + \sqrt{\frac{x}{\epsilon^2} + 1}\right\}^{1/n}$
207	GSB4	
208	1/X	
209	-	
210	RTN	
211	*LBL4	subroutine to calculate:
212	RCLE	
213	1/X	$(\ )^{1/n} \rightarrow R_x \rightarrow R_y$
214	Y^X	
215	ENT↑	
216	RTN	
217	*LBL8	subroutine to calculate:
218	RCL1	
219	×	$s_q = 2 \sin\left(\frac{\pi q}{n}\right) \rightarrow R_X$
220	2	
221	→R	
222	STOE	$c_q = 2 \cos\left(\frac{\pi q}{n}\right) \rightarrow R_E$
223	R↓	
224	RTN	NOTE TRIG MODE

LABELS					FLAGS
A filter order	B R_T	C Butterworth coefficients	D $-\epsilon_{dB}$ Cheb coefficients	E -3dB Cheb coefficients	0 -3dB Cheb
a	b	c	d	e	1 Chebyshev
0 recursion loop setup	1 recursion loop start	2 Chebyshev -3dB jump	3 print & space	4 $(\ )^{1/n}$	2 odd number
5 store g_{r+1}	6 SF2 if n odd	7 F(x)	8 S_q & C_q	9	3

SET STATUS			
FLAGS	ON	OFF	
0		■	
1		■	
2		■	
3			
TRIG	DEG	GRAD	RAD ■
DISP	FIX	SCI	ENG ■ n 5

PROGRAM 2-4 NORMALIZED LOWPASS TO BANDSTOP, LOWPASS, OR HIGHPASS LC LADDER TRANSFORMATIONS.

Program Description and Equations Used

This program transforms the normalized lowpass coefficients (1 ohm, 1 radian/sec) into the frequency and impedance scaled lowpass, highpass, or bandstop topologies. The normalized lowpass coefficients are obtained from register storage and either must be loaded by the user (for other than Butterworth and Chebyshev filters), or are generated and stored by Program 2-3 for the Butterworth and Chebyshev approximations.

Every linear, passive, lumped, time-invariant, bilateral electrical network has a dual topology. LC filters are a member of this class of networks; hence, two electrically equivalent networks can be formed from the transformation or scaling of the normalized lowpass structure. These two forms are designated as form 1, and form 2 in the program. Having two forms available provides the designer some relief from awkward component values, or the opportunity to choose the minimum inductor topology.

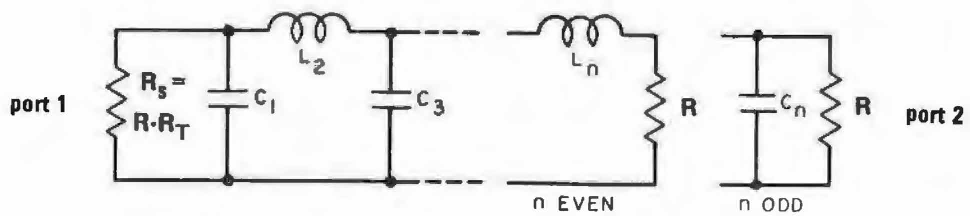
The program is separated into three parts which share common subroutines. These sections are 1) de-normalization parameter input (bandwidth, termination resistance level, and center frequency), 2) bandstop denormalization and transformation, and 3) lowpass and highpass denormalization and transformation. In analytical form, these transformations are discussed next.

Lowpass filters: No transformation is necessary for converting the normalized lowpass to the un-normalized lowpass filters. The normalized lowpass values need only be scaled to the desired operating impedance level and cutoff frequency. The object of the scaling procedure is to end up with filter elements that have the same impedance ratios to the termination resistance at the cutoff frequency as the normalized filter has at 1 radian/second to 1 ohm. The mechanics of this scaling procedure are:

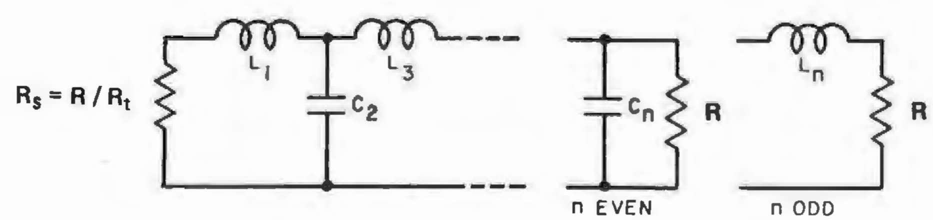
$$\text{L, scaled} = (\text{L, normalized}) \cdot (R/(2\pi \cdot BW)) \tag{2-4.1}$$

$$\text{C, scaled} = (\text{C, normalized}) \,/\, (2\pi \cdot BW \cdot R) \tag{2-4.2}$$

The normalized L's and C's are equal to the g's from Program 2-3, and BW and R represent the cutoff frequency in Hz and the load resistance in ohms respectively. Figure 2-4.1 shows the two forms of the lowpass filter; either port can be designated as input, i.e., the input voltage source can go in series with either termination resistor.



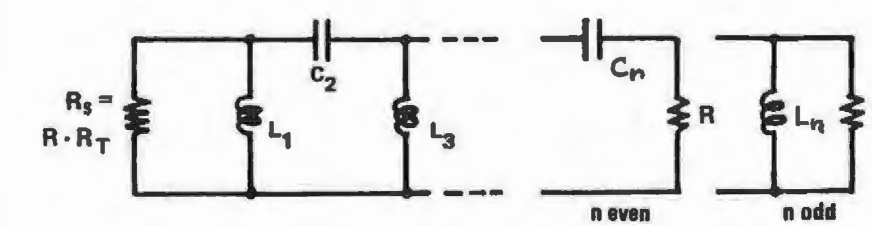
FORM 1



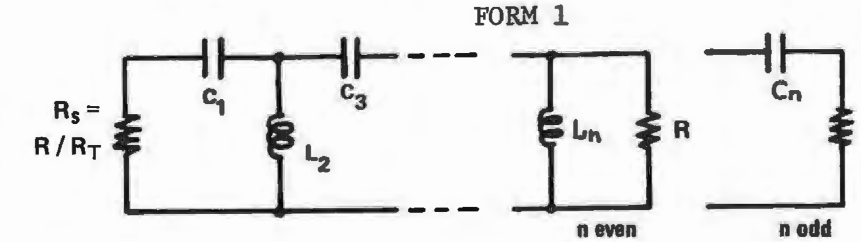
FORM 2

Figure 2-4.1 Two forms of lowpass filter.

Highpass filters: The highpass transformation is accomplished by replacing s by 1/s. Since sinusoidal frequencies are of primary interest, s may be replaced by $j\omega$, or 1/s by $-j/\omega$. Conceptually, this operation is equivalent to replacing each normalized lowpass capacitor with an inductor and vice-versa. The normalized values of the highpass elements are the reciprocals of the lowpass values, i.e., the g's calculated in Program 2-3 become 1/g's when converted to normalized highpass coefficients. Fig. 2-4.2 shows the two forms of the highpass filter, and the element values are calculated using Eqs. (2-4.1) and (2-4.2) with the normalized highpass coefficients. Either port can be designated as the input as in the lowpass case (or in any other passive LC case).



FORM 1



FORM 2

Figure 2-4.2 Two forms of highpass filter.

The highpass transformation may also be applied analytically; for example, the transformation is applied to the Butterworth normalized lowpass magnitude response equation (Eq. (2-4.3)) to convert it to the normalized highpass form (Eq. (2-4.4)).

$$|A(\omega)|_{LP} = \frac{1}{\sqrt{1 + \omega^{2n}}} \tag{2-4.3}$$

$$|A(\omega)|_{HP} = \frac{\omega^{n}}{\sqrt{1 + \omega^{2n}}} \tag{2-4.4}$$

For more information, see Weinberg [53]. Blinchikoff and Zverev [8] also has an excellent discussion of transformations both conventional as used herein, and unconventional to preserve LP transient characteristics.

Bandpass filters: The bandpass filter is a combination of a highpass and a lowpass filter. The loaded Q, Q_L, of the filter is a measure of the separation between the highpass and lowpass portions. To accomplish the transformation from normalized lowpass to un-normalized bandpass, s in the normalized lowpass expression is replaced by the function of s shown in Eq. (2-4.5).

$$s \Rightarrow Q_L \left\{ \frac{s}{\omega_o} + \frac{\omega_o}{s} \right\} \tag{2-4.5}$$

$$Q_L = \frac{f_o}{BW} \tag{2-4.6}$$

Where f_o and BW are the center frequency and bandwidth in hertz.

Conceptually, the lowpass elements are replaced with new elements that exhibit the same impedance behavior at the bandpass filter center frequency as did the original elements at dc. Ideal inductors have zero reactance at dc, and are replaced with series resonant tank circuits which resonate at the bandpass filter center frequency, f_o. Ideal (lossless) series tank circuits have zero reactance at resonance. Likewise, each lowpass capacitor is replaced with a parallel resonant tank circuit which resonates at the bandpass filter center frequency. When the loaded Q is greater than 10 or so, the bandpass filter is called narrowband. In this case, other tank circuits can be synthesized to approximate the impedance behavior of the series and parallel resonant tank circuits for frequencies within the vicinity of the passband. Bandpass filters and narrowband transformations are discussed in Programs 2-5, 2-6, and 2-11.

Bandstop filters: The bandstop transformation is the reciprocal of the bandpass transformation, and is analogous to the lowpass-highpass transformation. Highpass filters are actually bandstop filters which have zero center frequency. To accomplish the bandstop transformation, s is replaced by:

$$s \Rightarrow \frac{1}{Q_L\left\{\frac{s}{\omega_o} + \frac{\omega_o}{s}\right\}} \tag{2-4.7}$$

Conceptually the bandstop transformation is accomplished by designing a highpass filter whose cutoff frequency equals the bandwidth of the desired bandstop filter. Each shunt inductor in the highpass filter is series resonated with a capacitor at the desired center frequency of the filter. Likewise, each series capacitor is parallel resonated with an inductor at the desired filter center frequency.

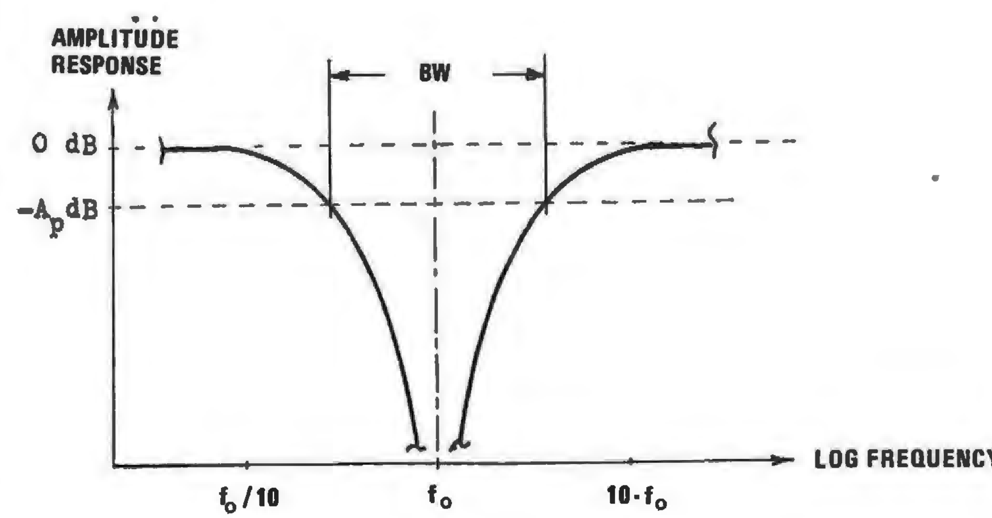


Figure 2-4.1 Bandstop filter parameters.

If g_1, g_2, ... , g_n are the normalized lowpass coefficients and R_T is the normalized termination resistance, then one form of the bandstop filter is shown by Fig. 2-4.2 .

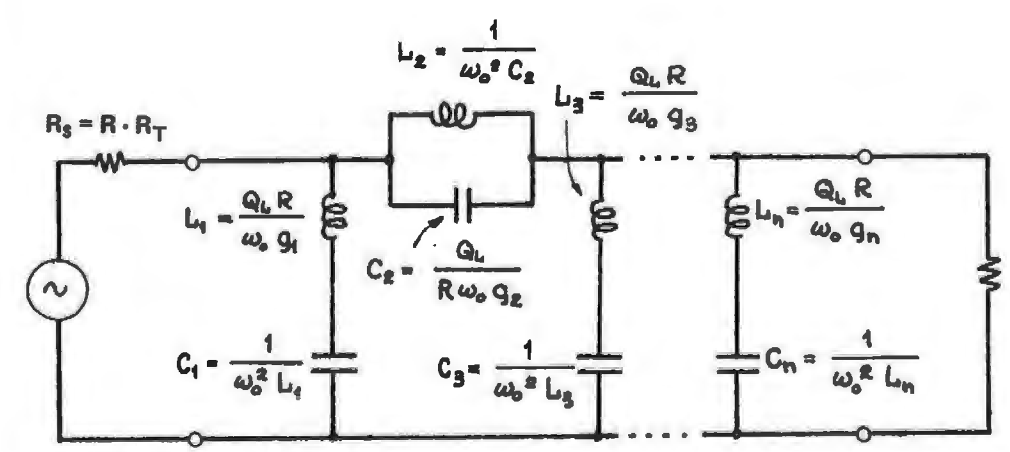


Figure 2-4.2 Bandstop filter form 1 (program output heading "21"), odd order filter shown; even order filter lacks last series tank circuit.

The other form of this filter is the dual of the first:

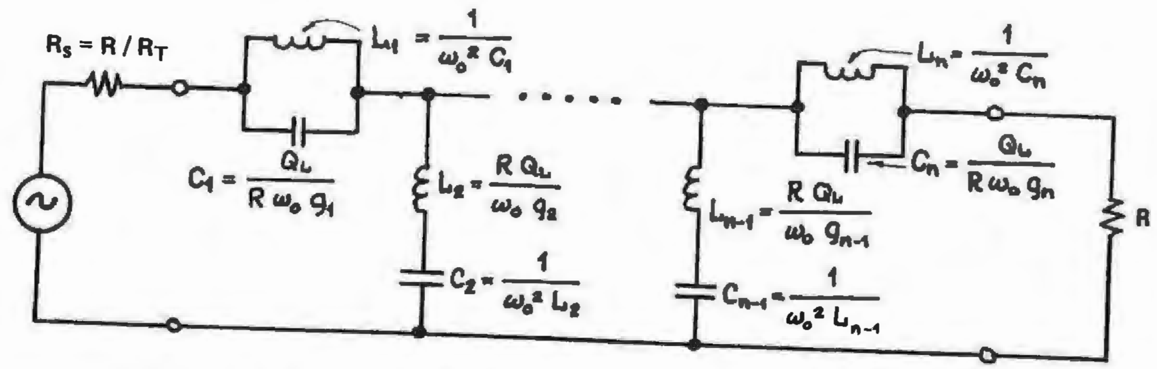


Figure 2-4.3 Bandstop form 2 (program output heading "22"), odd order filter shown; even order filter lacks last parallel tank circuit.

The program calculates both forms of these bandstop filters.

Filter physical realizability. The preceding transformations are used by this program and result in LC network schematics that will produce the desired response. Not all LC networks that can be drawn on paper as schematics are physically realizable. For example, a network branch consisting of a 1 μF capacitor in series with a 10 nh inductor would be nearly impossible to realize since the self inductance of the capacitor is much larger than the total required inductance. Table 2-4.1 is a reproduction of Table 7.1 from White [56], and shows the degree of physical realizability of lowpass and highpass filters. The physical realizability of a filter is assigned one of four possible scores. These scores are defined as follows:

Readily realizable (R): $1\ \mu h \le L \le 1\ h$
$5\ pF \le C \le 1\ \mu F$

Practical (P): $200\ nh \le L \le 10\ h$
$2\ pF \le C \le 10\ \mu F$

Marginally practical (M): $50\ nh \le L \le 100\ h$
$0.5\ pF \le C \le 500\ \mu F$

Impractical (I): All element values that lie outside the marginal range, i.e.,

$L < 50$ nh
$L > 100$ h
$C < 0.5$ pF
$C > 500$ μF

The table headings are meant to indicate ranges of filter cutoff frequency and termination impedance level. These ranges are defined as follows:

Frequency,

f_o = 10 Hz implies: 3 Hz $\le f_o <$ 30 Hz
f_o = 100 Hz implies: 30 Hz $\le f_o <$ 300 Hz
f_o = 1 kHz implies: 300 Hz $\le f_o <$ 3 kHz
f_o = 10 kHz implies: 3 kHz $\le f_o <$ 30 kHz
f_o = 100 kHz implies: 30 kHz $\le f_o <$ 300 kHz
f_o = 1 MHz implies: 300 kHz $\le f_o <$ 3 MHz
f_o = 10 MHz implies: 3 MHz $\le f_o <$ 30 MHz
f_o = 100 MHz implies: 30 MHz $\le f_o <$ 300 MHz

At frequencies above 300 MHz, lumped element filters are generally replaced with transmission line type filters.

Impedance Level (source and load resistances equal)

R = 3 ohms implies: $1 \le R < 10$ (power filters)
R = 50 ohms implies: $10 \le R < 150$
R = 500 ohms implies: $150 \le R < 2.5k$
R = 10k ohms implies: $2.5k \le R < 50k$

Table 2-4.1 Physical realizability of lowpass and highpass filters.

R in ohms	Cutoff Frequency, f_c							
	10 Hz	100 Hz	1 kHz	10 kHz	100 kHz	1 MHz	10 MHz	100 MHz
3	I	M	M	P	R	P	M	I
50	M	M	M	R	R	R	R	M
500	M	P	R	R	R	R	R	R
10k	I	M	P	R	R	R	P	I

Courtesy, Don White Consultants, Inc.

Bandstop filter physical realizability must include one additional parameter, the loaded Q of the filter, Q_L. As Q_L becomes higher (filter becomes more narrow) the separation in element value between the series tank elements and the parallel tank elements increases as Q_L . Table 2-4.2 is adapted from Table 7.2 in White and assigns realizability scores to bandstop (and bandpass) filters. The loaded Q ranges are defined as follows:

Loaded Q (Q_L), for bandpass and bandstop,

$$Q_L = 5 \text{ implies: } 3 \leq Q_L < 10$$
$$Q_L = 15 \text{ implies: } 10 \leq Q_L < 30$$
$$Q_L = 50 \text{ implies: } 30 \leq Q_L \leq 100$$

Table 2-4.2 Physical realizability of bandstop filters.

Filter Prototype	$f_o = 1$ kHz									$f_o = 10$ kHz								
	$Q_L = 5$			$Q_L = 15$			$Q_L = 50$			$Q_L = 5$			$Q_L = 15$			$Q_L = 50$		
Type	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K
1st	I	P	P	I	I	I	I	I	I	M	R	P	I	M	P	I	M	P
2nd	I	P	P	I	I	I	I	I	I	M	R	P	I	M	I	I	M	P
Filter Prototype	$f_o = 100$ kHz									$f_o = 1$ MHz								
	$Q_L = 5$			$Q_L = 15$			$Q_L = 50$			$Q_L = 5$			$Q_L = 15$			$Q_L = 50$		
Type	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K
1st	P	R	R	P	P	I	M	M	I	P	R	P	P	P	I	M	P	I
2nd	P	R	R	P	P	I	M	M	I	P	R	P	P	P	I	M	M	I
Filter Prototype	$f_o = 10$ MHz									$f_o = 100$ MHz								
	$Q_L = 5$			$Q_L = 15$			$Q_L = 50$			$Q_L = 5$			$Q_L = 15$			$Q_L = 50$		
Type	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K
1st	M	P	M	I	P	I	I	I	I	I	I	I	I	I	I	I	I	I
2nd	M	P	M	I	P	I	I	I	I	I	I	I	I	I	I	I	I	I

Courtesy Don White Consultants Inc.

As the loaded Q increases, the element value spread can become unmanageable. This problem can be reduced by using narrowband transformations which are used in Programs 2-5 and 2-6 for the bandpass case. Narrowband transformation schematics for the bandstop case may be found on p. 217 of the ITT handbook [44]. The concept of coupling and narrowband transformations was introduced by Milton Dishal [21], and expanded by Seymour Cohn [16] for the bandpass case.

BANDSTOP, LOWPASS, OR HIGHPASS TRANSFORMATIONS				
Lowpass Type 1	Lowpass Type 2	Highpass Type 1	Highpass Type 2	
Center Frequency	Bandwidth, Cutoff freq	Termination Resistance	Bandstop Type 1	Bandstop Type 2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	For lowpass filter component values:			
	a) load cutoff frequency in hertz	f_{cutoff}	[B]	
	b) load termination resistance in ohms	R	[C]	
	c) for type 1 filter (capacitor first)*		[f] [A]	R_s
				C_1
				L_2
				:
				C_n or L_n
				R
	d) for type 2 filter (inductor first)*		[f] [B]	R_s
				L_1
				C_2
				:
				L_n or C_n
				R
3	For highpass filter component values:			
	a) load cutoff frequency in hertz		[B]	
	b) load termination resistance in ohms		[C]	
	c) for type 1 filter (inductor first)*		[f] [C]	R_s
				L_1
				C_2
				:
				C_n or L_n
				R

BANDSTOP, LOWPASS, OR HIGHPASS TRANSFORMATIONS
CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
3	Highpass component values continued			
	b) for type 2 filter (capacitor first)*		[f] [D]	R_s
				C_1
				L_2
				:
				L_n or C_n
				R
4	For bandstop filter component values:			
	a) load filter center frequency in hertz	f_o	[A]	
	b) load filter bandwidth in hertz	BW	[B]	
	c) load termination resistance in ohms	R	[C]	
	d) for type 1 filter (series tank first)*		[D]	R_s
				C_1**
				L_1
				C_2
				L_2
				:
				C_n
				L_n
				R
	e) for type 2 filter (parallel tank first)*		[E]	R_s
				C_1
				L_1
				:
				C_n
				L_n
				R

NOTES:

* All capacitor values are in farads, all inductor values are in henries and all resistor values are in ohms.

** In all section 2 programs where resonant tank components are printed, the capacitor is always printed first.

Example 2-4.1 Singly terminated lowpass filter

A maximally flat (Butterworth) lowpass filter must pass a 1 kHz signal with 1 dB or less attenuation relative to the filter response at dc, and must reject a 12 kHz signal by at least 75 dB. Program 2-1 is used to predict the required filter order, and -3 dB cutoff frequency (Butterworth cutoff frequency) with $Ap_{dB} = 1$ dB, $As_{dB} = 75$ dB, and $\lambda = 12$. A filter order of 3.75 is calculated, which is rounded to the next largest integer, 4. Re-entering the program with $As_{dB} = 3$ and $n = 4$, yields $\lambda = 1.183301$, which means the 3 dB cutoff frequency is (1000)(1.183301) = 1183.301 Hz.

Next, Program 2-3 is loaded to obtain the normalized lowpass coefficients for a singly terminated 4th order Butterworth filter. The coefficients are automatically stored for use by this program.

Load this program, load the above cutoff frequency, and select an operating impedance level from Table 2-4.1 An impedance level of 500 ohms will result in a readily realizable filter. Both the type 1 and type 2 topologies can be calculated and the most attractive one selected. The HP-97 printout for the above operations is shown on the next page.

Programs 3-1 and 3-2 can be used to design the inductors needed for this design. If an active filter approach can be considered, see Program 2-9 for a lowpass active filter design.

HP-97 printout for Example 2-4.1, lowpass filter design.

Load Program 2-1 to calculate required filter order:

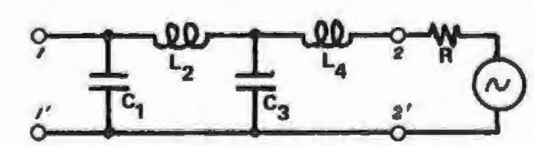
```
             GSBa   select Butterworth
     1.00    GSBA   load ApdB
    75.00    GSBB   load AsdB
    12.00    GSBC   load λ, and calculate n, the filter order
     3.75    ***    n (output)

     3.00    GSBB   load new AsdB
     4.00    GSBC   load integral filter order, n, and calculate λ
  1.183301   ***    λ (output)
```

Load Program 2-3 to generate and store the normalized lowpass coefs.

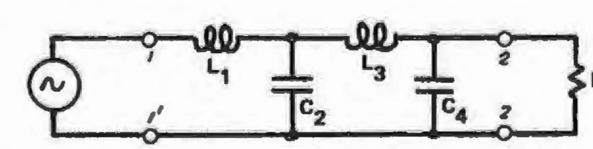
```
        4.   GSBA   load filter order
     1.+05   GSBB   load termination resistance desired
             GSBC   calculate Butterworth coefficients
1.00000+09   ***    RT  (normalized)

1.53073+00   ***    g1  )
1.57716+00   ***    g2  )  lowpass normalized coefficients
1.08239+00   ***    g3  )
382.683-03   ***    g4  )

1.00000+00   ***    R  (normalized)
```

Load this program (Program 2-4) to obtain un-normalized filter.

```
  1181.301   GSBE   load un-normalized cutoff frequency
      500.   GSBC   load termination resistance, R
             GSBa   calculate type 1 lowpass filter (capacitor first)

       31.          lowpass type 1 output code

  500.0+03   ***    Rs  (open circuit)

  412.5-09   ***    C1
  106.2-03   ***    L2
  291.7-09   ***    C3
  25.78-03   ***    L4

  500.0+00   ***    R
```




```
             GSBb   calculate type 2 lowpass filter (inductor first)

       32.          lowpass type 2 output code

  500.0-09   ***    Rs  (short circuit)

  103.1-03   ***    L1
  425.0-09   ***    C2
  72.91-03   ***    L3
  103.1-09   ***    C4

  500.0+00   ***    R
```



Example 2-4.2 Doubly terminated highpass filter

A highpass filter is required to keep the signal from a local CB transmitter from causing cross modulation interference in the tuner of a TV set. The filter will be placed in series with the 300 ohm balanced line from the antenna to the TV set, hence, the filter will be designed for a 300 ohm terminating impedance level. The filter must pass the TV spectrum which starts at 54 MHz, but must reject the CB radio band at 27 MHz. One dB of ripple is allowed across the TV spectrum, and at least 60 dB rejection is needed at the CB band frequencies. Because of the allowed ripple, a Chebyshev filter will be used. Program 2-1 calculates a minimum filter order of 7 as shown below along with the rest of the HP-97 printout for this design:

Load Program 2-1 to obtain minimum filter order required:

```
        GSBe   select Chebyshev
   1.00 GSBA   load ApdB
  60.00 GSBB   load AsdB
   2.00 GSBD   load λ and calculate filter order, n
   6.22  ***   n (output)

   7.00 GSBC   load integral filter order, n
   1.78  ***   λ where filter is 60 dB down (54/1.78 = 30.3)

   2.00 GSBE   load λ and calculate AsdB
  66.15  ***   AsdB at 27 MHz
```

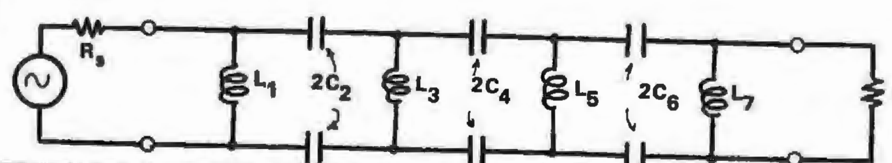
Load Program 2-3 to generate and store the normalized lowpass coefficients:

```
         7. GSBA   load filter order
         1. GSBB   load termination resistance ratio
         1. GSBD   load Chebyshev passband ripple in dB and start
1.01721+00   ***   normalized -3 dB frequency (output)

1.00000+00   ***   RT (normalized)

2.16656+00   ***   g1 \
1.11151+00   ***   g2  |
3.09364+00   ***   g3  |
1.17352+00   ***   g4  } normalized lowpass coefficients
3.09364+00   ***   g5  |
1.11151+00   ***   g6  |
2.16656+00   ***   g7 /

1.00000+00   ***   R (normalized)
```

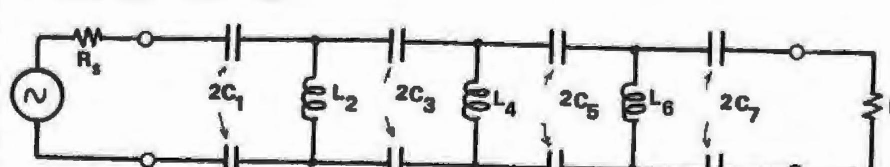

This highpass example is for a balanced structure filter, and the program output is for an unbalanced structure (one side common). To convert the unbalanced structure to the balanced structure, capacitors are placed in each side of the filter, and their equivalent impedance is one-half the unbalanced value (twice the capacity).

Load this program, Program 2-4, to obtain the un-normalized filter:

```
  54.+06 GSBB   load cutoff frequency
    300. GSBC   load denormalization resistance
         GSBc   calculate type 1 highpass filter values

     41.        highpass type 1 output code (inductor first)

300.0+00  ***   Rs

406.1-09  ***   L1
8.835-12  ***   C2
265.8-09  ***   L3
8.372-12  ***   C4
265.8-09  ***   L5
8.835-12  ***   C6
406.1-09  ***   L7

300.0+00  ***   R
```




```
         GSBa   calculate type 2 highpass filter values

     42.        highpass type 2 output code (capacitor first)

300.0+00  ***   Rs

4.535-12  ***   C1
795.5-09  ***   L2
3.176-12  ***   C3
753.5-09  ***   L4
3.176-12  ***   C5
795.5-09  ***   L6
4.535-12  ***   C7

300.0+00  ***   R
```



Programs 3-5 and 3-6 can aid in the aircore inductor designs needed here.

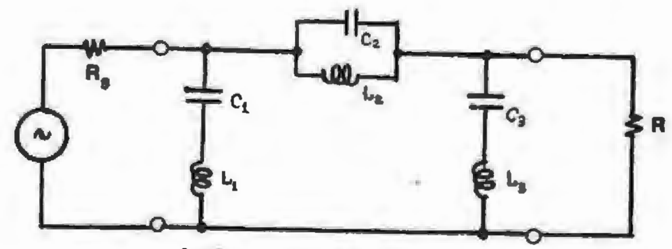
Example 2-4.3 Bandstop filter

Consider implementing the filter cited in the previous example as a bandstop filter rather than as a highpass filter. The stopband required is from 26 MHz to 27 MHz. The center frequency of a bandstop (and bandpass) filter is the geometric mean of any two equal attenuation frequencies (this relationship does not hold for narrowband bandpass transformations for frequencies outside the passband). The center frequency of this bandstop filter is then 26.4953 MHz. If the upper -1 dB point is 54 MHz, then the lower -1 dB point is $(26.4953 \text{ MHz})^2/(54 \text{ MHz}) = 13$ MHz. The normalized frequency multiplier, λ, is the ratio between the passband and the stopband, or $\lambda = (54 - 13)/(27 - 26) = 41$. From Program 2-1, the filter order that meets these requirements is 2. Even ordered Chebyshev filters do not have equal termination resistance levels, and this filter is to be placed in a 300 ohm system (equally terminated). To satisfy all requirements including equal termination, a third order bandstop filter will be designed. The HP-97 printout for this filter design follows.

Load Program 2-1 to calculate the minimum filter order required:

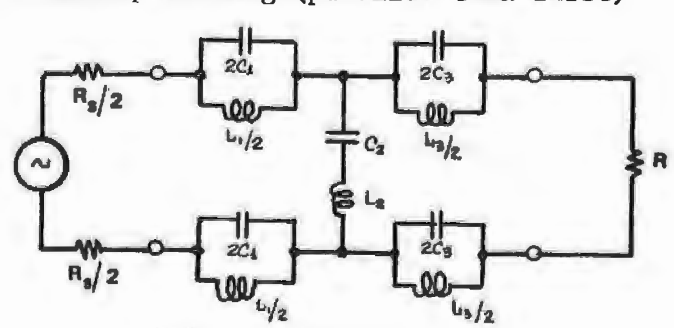
```
          GSBb   select Chebyshev
    1.00  GSBA   load Ap_dB
   60.00  GSBE   load As_dB
   41.00  GSBD   load λ and calculate filter order, n
    1.86  ***    minimum n to meet requirements (use n = 2 min)

    3.00  GSBC   load n desired and calculate λ for As_dB = 60 dB
    7.92  ***    λ

           1/x }
   41.00    x  } form 1/λ
    5.18  ***    stopband bandwidth (MHz)
 26.4953  GSBd   enter center frequency (MHz) and calculate:
   29.21  ***    upper stopband edge (MHz)
   24.03  ***    lower stopband edge (MHz)
```

3. GSBA		load filter order
1. GSBB		load termination resistance ratio desired
1. GSBD		load Chebyshev passband ripple in dB and start
1.09487+00	***	λ for 3 dB bandwidth (output)
1.00000+00	***	R_T (normalized)
2.02359+00	***	normalized lowpass coefficients
994.102-03	***	
2.02359+00	***	
1.00000+00	***	R (normalized)
26.4553+06 GSBA		load filter center frequency
41.+06 GSBB		load filter bandwidth
300. GSBC		load de-normalization resistance level
GSBD		calculate type 1 bandstop
21.		type 1 bandstop heading (series tank first)
300.0+00	***	R_s
62.70-12	***	C_1
575.5-09	***	L_1
13.02-12	***	C_2
2.772-06	***	L_2
62.70-12	***	C_3
575.5-09	***	L_3
300.0+00	***	R



unbalanced structure

GSBE		calculate type 2 bandstop
22.		type 2 bandstop heading (parallel tank first)
300.0+00	***	R_s
6.394-12	***	C_1
5.643-06	***	L_1
30.80-12	***	C_2
1.171-06	***	L_2
6.394-12	***	C_3
5.643-06	***	L_3
300.0+00	***	R



balanced structure

Program Listing I

Key	Comment
*LBLA	LOAD CENTER FREQUENCY
Pi	
ENT↑	
+	$2\pi f_o \to R0$
x	
STO0	
RTN	
*LBLB	LOAD BANDWIDTH OR CUTOFF FREQ
Pi	
ENT↑	
+	$2\pi BW \to R1$
x	
STO1	
RTN	
*LBLC	LOAD DENORMALIZATION RESIST
STO2	$R \to R2$
RTN	
*LBLD	BANDSTOP TYPE 1 ROUTINE
SPC	
2	
1	print heading "21"
PRTX	
CF1	indicate type 1 topology
GTO0	
*LBLE	BANDSTOP TYPE 2 ROUTINE
SPC	
2	
2	print heading "22"
PRTX	
SF1	indicate type 2 topology
*LBL0	
SF2	calculate and print R_s
GSB2	
RCL2	calculate and store:
RCL1	
x	$\frac{R \cdot BW}{\omega_o^2} = \frac{R}{\omega_o \cdot Q_L} \to R4$
RCL0	
X²	
÷	
STO4	
RCL2	
X²	$\frac{1}{R \cdot \omega_o \cdot Q_L} \to R5$
÷	
STO5	
CLX	
STO7	initialize index registers
9	
STOI	
*LBL1	bandstop calculation loop
GSB9	increment indices and
X>Y?	test for loop exit
GTO4	
SPC	
RCLi	recall g_k
RCL4	recall $R/(\omega_o \cdot Q_L)$
CF0	set print order for type 1
F1?	test for type 1 filter
GTO6	
CLX	
RCL5	substitute $1/(R \cdot \omega_o \cdot Q_L)$ in R_x
SF0	set print order for type 2
*LBL6	
GSB8	gosub elt calculation & print
GSB9	increment indices and
X>Y?	test for loop exit
GTO4	
SPC	
RCLi	recall g_k
RCL5	recall $1/(R \cdot \omega_o \cdot Q_L)$
SF0	set print order for type 1
F1?	test for type 1 filter
GTO6	
CLX	
RCL4	substitute $R/(\omega_o \cdot Q_L)$ in R_x
CF0	
*LBL6	
GSB8	gosub elt calculation & print
GTO1	goto loop start
*LBL8	element calculation & print
x	form and store $g_k \cdot R_x \to R8$
STO8	
F0?	if flag 0 print R8
PRTX	
RCL0	calculate mating
X²	resonant element:
x	$C, L = \frac{1}{\omega_o^2 \cdot (L, C)}$
1/X	
PRTX	
F0?	if flag 0, return to
RTN	main program
RCL8	recall and print R8
PRTX	
RTN	return to main program
*LBLa	LOWPASS TYPE I ROUTINE
SPC	
3	
1	print heading "31"
PRTX	
CF0	indicate lowpass filter
CF1	indicate type 1 filter
GSB7	calculate some constants
GTO2	goto output routine

REGISTERS

0	1	2	3	4	5	6	7	8	9
$2\pi f_c$	$2\pi BW$	R		scratch	scratch	n	k		

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}

A	B	C	D	E	I
g_{11}	g_{12}	g_{13}	R_T		index

2-4

Program Listing II

```
103  *LBLb   LOWPASS TYPE 2 ROUTINE
104   SPC
105     3    print heading "32"
106     2
107  PRTX
108   CF0    indicate lowpass filter
109   SF1    indicate type 2 filter
110  GSB7    compute LP type 1 constants
111  RCL2
112    X²    change to LP type 2 constants
113  ST÷4
114  ST×5
115  GTO2    goto output routine
116  *LBLd   HIGHPASS TYPE 2 ROUTINE
117   SPC
118     4    print heading "42"
119     2
120  PRTX
121   SF0    indicate highpass
122   SF1    indicate type 2
123  GSB7    compute LP type 1 constants
124  GTO2    goto output routine
125  *LBLc   HIGHPASS TYPE 1 ROUTINE
126   SPC
127     4    print heading "41"
128     1
129  PRTX
130   SF0    indicate highpass
131   CF1    indicate type 1
132  GSB7    calculate LP type 1 constants
133  RCL2
134    X²    change to HP type 1 constants
135  ST÷4
136  ST×5
137  *LBL2   LP & HP output routine
138   SPC    recall R_T
139  RCLD
140   F1?    if type 2 filter, form 1/R_T
141   1/X
142  RCL2
143     ×    calculate and print R_s
144  PRTX
145   F2?    test for return to bandstop
146   RTN
147   SPC
148  *LBL3   LP and HP output loop start
149  GSB9    increment indices and
150  X>Y?    test for loop exit
151  GTO4
152  RCLi    recall g_k
153   F0?    if highpass, form 1/g_k
154   1/X
155  RCL5    calculate and print
156     ×    first filter element
157  PRTX
158  GSB9    increment indices and
159  X>Y?    test for loop exit
160  GTO4
161  RCLi    recall g_k
162   F0?    if highpass, form 1/g_k
163   1/X
164  RCL4    calculate and print
165     ×    other type of filter element
166  PRTX
167  GTO3    goto loop start
168  *LBL4   recall and print port 2
169   SPC    termination resistance
170  RCL2
171  PRTX
172   SPC
173   SPC
174   RTN    return control to keyboard
175  *LBL7   subroutine to calc LP - 1
176  RCL2    calculate and store
177  RCL1    inductor scaling:
178     ÷    R/(2π.BW) → R4
179  STO4
180  RCL2    calculate and store
181    X²    capacitor scaling:
182     ÷    1/(2π.BW.R) → R5
183  STO5
184   CLX
185  STO7    initialize indices
186     9
187  STOI
188   RTN
189  *LBL9   incr indices and loop exit
190   EEX
191  ST+7
192  ISZI
193  RCL6
194  RCL7
195   RTN
```

LABELS					FLAGS	SET STATUS		
A load f_0	B load BW	C load R	D BS_1	E BS_2	0 Highpass	FLAGS	TRIG	DISP
a LP_1	b LP_2	c HP_1	d HP_2	e	1 Type 2	ON OFF		
0 calculate BS coef	1 BS loop rtn	2 calc R_T'	3 LP/HP loop rtn	4 print R	2 lbl 2 return	0 ■	DEG	USERS CHOICE FIX
5	6 Local loop destination	7 LP/HP coefs	8 bandstop output	9 Index incr & loop exit test	3	1 ■	GRAD	SCI
						2 ■	RAD	ENG
						3		n___

HP-67 suggested program changes. A print or R/S routine has not been provided, although register 9 and label "e" could have been used for this purpose. The reason for this omission is to preserve the heading format. Any program statements placed between a numeric entry and a print statement cause the printed format to be in the set status of the program; however, by placing the print statement directly after the numeric entry (see lines 20 through 22), "21" is printed without trailing zeros.

On the HP-67, the "print" statement causes the program halt for 5 seconds and a flashing decimal point. This situation slows program execution and may not be desirable. The HP-67 user may wish to have the program stop at the data output points. To cause the program to stop at these points, change the program as follows: Delete steps 019 - 022, 026 - 029, 095 - 098, 104 - 107, 117 - 120, and 126 - 129. Change the "print" statements to "R/S" statements at the following line numbers: 083, 088, 092, 144, 157, 166, and 171. To restart program execution after a program halt, execute a "R/S" from the keyboard.

Remember, when deleting steps from a program, always work from the back of the program forward. By observing this convention, the line numbers of steps not yet deleted will remain unaltered.

PROGRAM 2-5 NORMALIZED LOWPASS TO BANDPASS FILTER TRANSFORMATIONS, TYPES 1, 2, 6, AND 7.

Program Description and Equations Used

This program converts normalized lowpass filter element values to a set of four bandpass topologies [16], [21], [56]. The four topologies are shown in Fig. 2-5.1, and the parameter A_{ij} is defined by Eq. (2-5.1). Types 1 and 2 are exact transformations and will transform the lowpass response independent of the loaded filter Q (Eq. (2-4.7)). Types 6 and 7 of this program, and types 8, and 9 of Program 2-6 are narrowband approximations, and only provide accurate transformation results when the loaded Q is greater than 5, and preferably greater than 10.

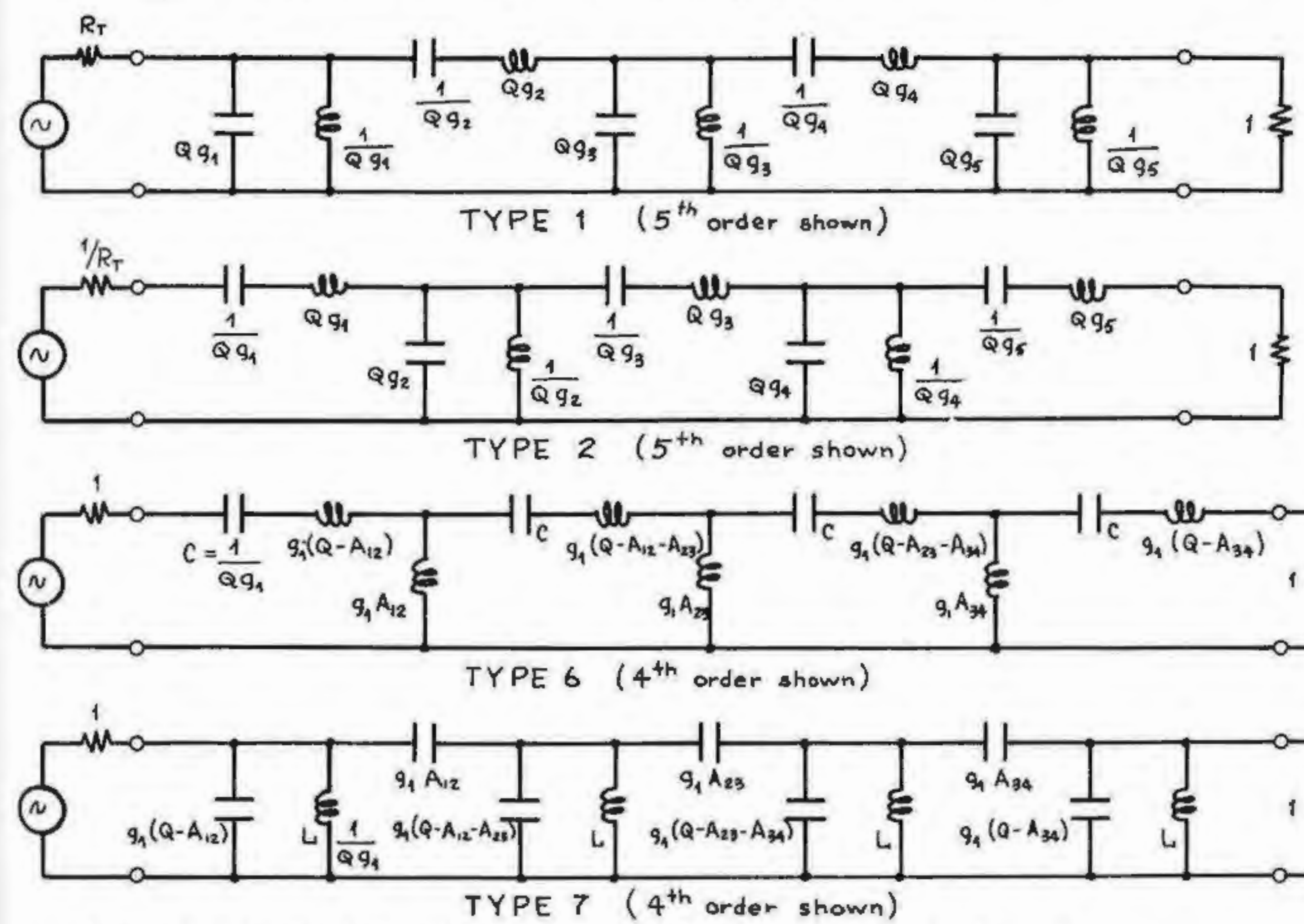


Figure 2-5.1 Bandpass filter topologies for types 1, 2, 6, & 7.

$$A_{ij} = (g_i \cdot g_j)^{-\frac{1}{2}} \tag{2-5.1}$$

Figure 2-5.2 is a reproduction of Table 7.2 in White [56] and is intended as a guide to the best suited filter topology for a particular application. The physical realizability of a filter topology is assigned one of four possible scores based upon element values. These scores are defined as follows:

Readily realizable (R):

$1\ \mu h \leq L \leq 1\ h$

$5\ pF \leq C \leq 1\ \mu F$

Practical (P):

$0.2\ \mu h \leq L \leq 10\ h$

$2.\ pF \leq C \leq 10\ \mu F$

Marginally practical (M):

$50\ nh \leq L \leq 100\ h$

$0.5\ pF \leq C \leq 500\ \mu F$

Impractical (I):

All element values that lie outside the range of marginal i.e.,

$L < 50\ nh$

$L > 100\ h$

$C < .5\ pF$

$C > 500\ \mu F$

The table headings are meant to indicate ranges of loaded Q, filter center frequency, and termination resistance level. These ranges are:

Frequency;

f_o = 10 Hz implies: 3 Hz $\leq f_o <$ 30 Hz

f_o = 100 Hz implies: 30 Hz $\leq f_o <$ 300 Hz

f_o = 1 kHz implies: 300 Hz $\leq f_o <$ 3 kHz

f_o = 10 kHz implies 3 kHz $\leq f_o <$ 30 kHz

f_o = 100 kHz implies: 30 kHz $\leq f_o <$ 300 kHz

f_o = 1 MHz implies: 300 kHz $\leq f_o <$ 3 MHz

f_o = 10 MHz implies: 3 MHz $\leq f_o <$ 30 MHz

f_o = 100 MHz implies: 30 MHz $\leq f_o <$ 300 MHz

At frequencies above 300 MHz, lumped element filters are generally replaced with transmission line type filters.

Loaded Q (Q_L), for bandpass and bandstop,

Q_L = 5 implies: $3 \leq Q_L < 10$

Q_L = 15 implies: $10 \leq Q_L < 30$

Q_L = 50 implies: $30 \leq Q_L \leq 100$

Impedance Level (source and load resistances equal)

R = 3 ohms implies: $1 \leq R < 10$ (power filters)

R = 50 ohms implies: $10 \leq R < 150$

R = 500 ohms implies: $150 \leq R < 2.5k$

R = 10k ohms implies: $2.5k \leq R < 50k$

Band-Pass Filter Prototype	f_o = 1 kHz									f_o = 10 kHz								
	$Q_L = 5$			$Q_L = 15$			$Q_L = 50$			$Q_L = 5$			$Q_L = 15$			$Q_L = 50$		
Type	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K
1st	I	P	P	I	I	I	I	I	I	M	R	P	I	M	P	I	M	P
2nd	I	P	P	I	I	I	I	I	I	M	R	P	I	M	I	I	M	P
3rd	I	M	I	I	I	I	I	I	I	M	P	M	I	P	P	I	M	I
4th	I	M	P	I	I	P	I	I	M	M	P	P	I	P	P	I	M	P
5th	P	P	P	P	I	P	P	P	I	R	R	P	R	P	P	R	P	M
6th	P	P	I	P	P	I	P	P	I	R	R	P	R	P	P	R	P	M
7th	I	M	P	I	I	P	I	I	M	M	P	R	I	P	R	I	M	P
8th	M	P	I	M	P	I	M	P	I	P	R	P	P	P	P	P	P	M
9th	I	M	P	I	I	P	I	I	M	M	P	P	I	P	P	I	M	P
10th	M	P	M	P	R	P	R	R	P	P	R	P	R	R	R	R	R	R
11th	M	P	P	M	P	P	M	M	P	P	R	R	P	R	R	M	P	R

Band-Pass Filter Prototype	f_o = 100 kHz									f_o = 1 MHz								
	$Q_L = 5$			$Q_L = 15$			$Q_L = 50$			$Q_L = 5$			$Q_L = 15$			$Q_L = 50$		
Type	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K
1st	P	R	R	P	P	I	M	M	I	P	R	P	P	P	I	M	P	I
2nd	P	R	R	P	P	I	M	M	I	P	R	P	P	P	I	M	M	I
3rd	P	R	P	P	R	P	M	P	P	R	R	P	R	R	R	M	P	I
4th	P	R	R	P	R	R	M	P	R	R	R	R	P	R	R	M	P	R
5th	R	R	P	R	R	P	R	R	P	R	R	P	R	R	M	R	P	I
6th	R	R	P	R	R	P	R	P	P	R	R	P	R	R	M	R	P	I
7th	P	R	R	P	R	R	M	P	R	R	R	R	P	R	R	M	P	R
8th	R	R	P	R	R	P	R	R	P	R	R	P	R	R	M	R	P	I
9th	P	R	R	P	R	R	M	P	R	R	R	R	P	R	R	M	P	R
10th	R	R	R	R	R	P	R	R	P	R	R	P	R	R	M	R	R	M
11th	R	R	R	R	R	R	P	R	R	R	R	R	R	R	R	R	R	R

Band-Pass Filter Prototype	f_o = 10 MHz									f_o = 100 MHz								
	$Q_L = 5$			$Q_L = 15$			$Q_L = 50$			$Q_L = 5$			$Q_L = 15$			$Q_L = 50$		
Type	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K	50	500	10K
1st	M	P	M	I	P	I	I	I	I	I	I	I	I	I	I	I	I	I
2nd	M	P	M	I	P	I	I	I	I	I	I	I	I	I	I	I	I	I
3rd	P	R	M	R	P	I	I	M	I	I	M	I	I	I	I	I	I	I
4th	M	R	R	I	P	R	I	M	R	I	M	R	I	I	P	I	I	M
5th	P	R	M	M	P	I	P	M	I	M	M	I	I	I	I	M	I	I
6th	R	R	M	P	P	I	P	M	I	P	M	I	M	I	I	M	I	I
7th	M	R	P	I	P	P	I	P	P	I	P	M	I	I	I	I	M	I
8th	R	R	M	R	P	I	P	M	I	P	M	I	P	I	I	M	I	I
9th	M	R	R	I	P	R	I	M	R	I	M	R	I	I	P	I	I	M
10th	P	R	I	P	M	I	P	M	I	P	M	I	M	I	I	M	I	I
11th	M	R	M	M	P	M	I	P	M	I	M	I	I	M	I	I	I	I

Fig. 2-5.2 Physical realizability of bandpass filters.

Courtesy Don White Consultants Inc.

To use the routines for types 6 through 9, the filter must have termination resistances as close to unity as possible. To achieve this result, a desired termination resistance level of 1.0 should be loaded into Program 2-3.

Of the filter types presented both in this program, and the accompanying program (types 1, 2, 6, 7, 8, 9, 10, and 11) only types 1, 2, 10, and 11 are exact transformations of the lowpass characteristic. All the remaining filter types are narrowband approximations, i.e., they will faithfully transform the lowpass characteristics within the passband and within a few octaves of the stopband. Types 6, 7, 8, and 9 do not have equal numbers of transmission zeros at both zero frequency and at infinite frequency. The result of this imbalance is to skew the filter response away from the frequency where the extra zeros exist. Figure 2-5.3 shows this occurrence.

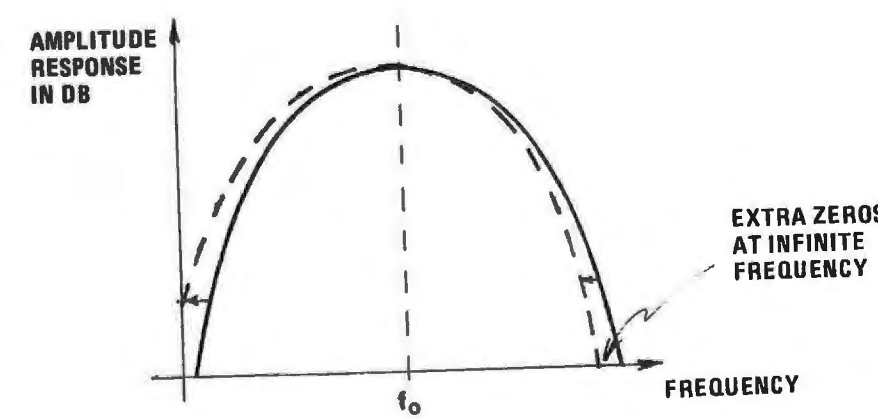


Figure 2-5.3 Bandpass filter response skewing due to extra transmission zeros at infinity.

One should not choose types 1, 2, 10, or 11 automatically. Types 1 and 2 may be difficult to realize in a narrowband application, and types 10 and 11 (also types 6 and 9) contain redundant inductors. Depending upon the frequency range and element values, these redundant inductors can be burdensome. As a guide, filters operating below 1 kHz may best be realized with an active filter (this subject is covered by other programs in this section); between 1 kHz and 100 kHz, the minimum inductor LC design should be considered and compared with active approaches; above 1 MHz the simplest LC topology should be sought to ease the tuning problem.

2-5 User Instructions

NORMALIZED LOWPASS TO BANDPASS TYPES 1, 2, 6, & 7				
load center frequency	load bandwidth	load termination resistance		load filter # & start

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Load center frequency in Hz	f_o	A	ω_o
3	Load bandwidth in Hz	BW	B	Q
4	Load termination resistance in ohms	R	C	R
5	Load filter type number and start	type	E	R
				tank C
				tank L
				cplg elt*
				tank C
				tank L
				cplg elt*
				tank C
				tank L
				⋮
	*The coupling element (L for type 6 or C for type 7) does not exist for types 1 or 2.			R_T
6	For another case goto steps 2 through 5 as applicable			

Example 2-5.1 Type 6 filter

A maximally flat passband (Butterworth) bandpass filter is to pass a 500 Hz band of frequencies centered around 10 kHz. In a bandpass (or bandstop) filter, the center frequency is the geometric mean of the upper and lower bandedge frequencies, i.e., $f_o = (9750 \cdot 10250)^{1/2}$, or $f_o = 9996.87$ Hz. The filter should reject by at least 30 dB those frequencies removed from the center frequency by more than 500 Hz. The required filter order is obtained from Program 2-1. Using this program, a minimum filter order of 5 is calculated given $As_{dB} = 3$, $Ap_{dB} = 30$, and $\lambda = 1000/500 = 2$. Program 2-3 is used to obtain the Butterworth normalized coefficients for use by this program.

The proper bandpass topology is selected from the table in Fig. 2-5.2 under the headings: $f_o = 10$ kHz, $Q_L = 10000/500 = 20$ (use $Q_L = 15$ column), and R = 50 to find that a type 6 is readily realizable, therefore a type 6 filter will be designed. The HP-97 printout for the above operations is shown below.

Load Program 2-1 to calculate minimum filter order:

```
        GSBa   select Butterworth
   3.00 GSBb   load ApdB
  30.00 GSBB   load AsdB
   2.00 GSBC   load λ and calculate n
   4.59  ***   filter. order, n (output)

   5.00 GSBD   load integral n and calculate λ
   2.00  ***   λ for given ApdB and AsdB

   2.00 GSBE   load λ and calculate AsdB
  32.05  ***   AsdB
```

Load Program 2-3 to calculate normalized LP Butterworth coefs:

```
          5. GSBA   load filter order
          1. GSBB   load desired termination resistance ratio
             GSBC   calculate Butterworth coefficients
1.00000+00    ***   RT (normalized port 1 termination resistor)

616.034-03    ***   g1 )
1.61803+00    ***   g2 )
2.00000+00    ***   g3 } normalized lowpass  coefficients
1.61803+00    ***   g4 )
616.034-03    ***   g5 )

1.00000+00    ***   R (normalized port 2 termination resistor)
```

Example 2-5.1, continued:

Load Program 2-5 (this program) and calculate type 6 elements.

```
  9996.87 GSBA  load center frequency
     500. GSBB  load bandwidth
      50. GSBC  load termination resistance
       6. GSBE  load filter type desired and start

50.000+00  ***  termination resistance

25.768-09  ***  C1
9.3443-03  ***  L1

491.97-06  ***  L12

25.768-09  ***  C2
9.0709-03  ***  L2

273.48-06  ***  L23

25.762-09  ***  C3
9.2894-03  ***  L3

273.48-06  ***  L34

25.768-09  ***  C4
9.0709-03  ***  L4

491.97-06  ***  L45

25.768-09  ***  C5
9.3443-03  ***  L5

50.000+00  ***  termination resistance
```

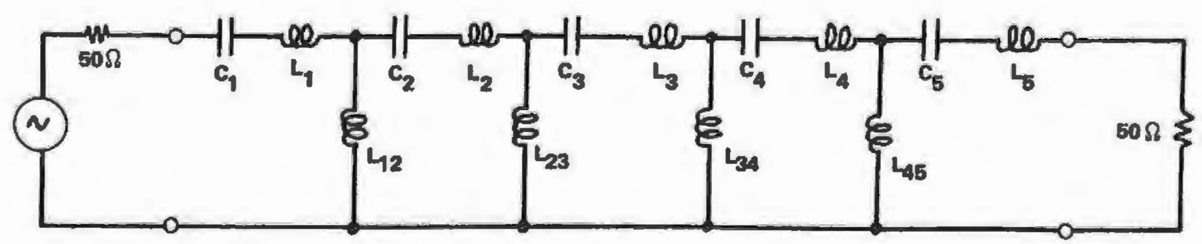


Figure 2-5.4 Type 6 bandpass filter schematic.

Type 6 tuning technique.* After the filter is designed, the inductors fabricated and adjusted, the capacitors obtained, and the filter constructed, the filter must be tuned. For series resonant tanks, such as in this filter and types 8 and 10, tuning is accomplished by decoupling individual tank circuits using open circuits.

Assume the inductors are wound on ferrite pot cores, and are the adjustable elements. Referring to Fig. 2-5.5, to tune L_1 temporarily open the circuit at "B" and tune L for series resonance of the L_1, L_{12}, C circuit at the center frequency of the filter, 9996.87 Hz in this case.

To tune L_2, L_{12}, L_{23}, and C, re-establish the connection at "B," and temporarily open the circuit at points "A" and "C." Tune L_2 for series resonance at the center frequency. Continue this procedure of opening adjacent tank circuits and tuning until all series resonant loops have been tuned to the filter center frequency.

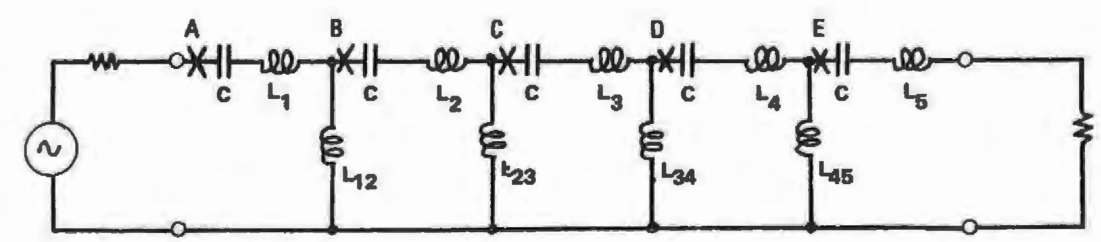


Figure 2-5.5 Type 6 filter showing circuit opens for tuning.

For information on ferrite pot core inductor design, see the Ferroxcube catalog "Linear Ferrite Materials and Components," and use Programs 3-1 and 3-2 to aid in the design of these inductors.

When designing the inductors, the designer must not allow the magnetic core excitation to drive the core near saturation. The voltage across an inductor is "Q" times the voltage across the series LC tank at

*For parallel tank filter tuning procedure, see the example in the type 8, 9, 10, and 11 transformations program.

resonance. With inductor Q's of 100 or better, inductor voltages can be large with respect to the voltage across the filter. The voltage across a filter element at center frequency is approximately $I_{in} \cdot X_{element}$ where I_{in} is the filter input current and $X_{element}$ is the element reactance.

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD CENTER FREQUENCY
002	Pi	
003	ENT↑	form and store:
004	+	
005	×	$2\pi f_o \to R0$
006	STO0	
007	RTN	
008	*LBLB	LOAD BANDWIDTH
009	Pi	
010	ENT↑	
011	+	form and store:
012	×	
013	RCL0	$Q = \frac{2\pi f_o}{2\pi BW} \to R_1$
014	X⇄Y	
015	÷	
016	STO1	
017	RTN	
018	*LBLC	LOAD TERMINATION
019	STO2	RESISTANCE
020	RTN	
021	*LBLE	LOAD FILTER TYPE (1,2,6,7)
022	STOI	
023	8	generate "ERROR" if other
024	X≤Y?	than filter types 1, 2,
025	GTOa	6, or 7 loaded.
026	RCLI	
027	3	"ERROR" is generated by
028	X=Y?	calling unused label (a)
029	GTOa	
030	RCLI	
031	4	
032	X=Y?	
033	GTOa	
034	RCLI	
035	5	
036	X=Y?	
037	GTOa	
038	RCLI	calculate indirect label
039	2	corresponding to desired
040	÷	filter type
041	STOI	
042	INT	
043	X⇄I	
044	FRC	
045	SF0	set flag 0 if types 1
046	X=0?	or 7 are entered otherwise
047	CF0	clear flag 0
048	GTOi	
049	*LBL0	type 1 calculation start
050	*LBL1	type 2 calculation start
051	GSB9	initialize flags & regs
052	*LBL2	types 1 & 2 loop start
053	RCL7	
054	2	set flag 2 if branch
055	÷	number is even where
056	FRC	k is the branch number
057	X=0?	
058	SF2	
059	RCLi	
060	RCL1	form $Q \cdot g_i$
061	×	
062	F2?	if branch even, form $1/Q \cdot g_i$
063	1/X	
064	STOC	calculate and print
065	F0?	branch capacitor
066	GSB8	
067	F1?	
068	GSB7	
069	RCLC	
070	F0?	calculate and print
071	GSB7	branch inductor
072	F1?	
073	GSB8	
074	SPC	
075	DSZI	
076	EEX	increment indices
077	ST+7	
078	RCL6	
079	RCL7	test for loop exit
080	X≤Y?	
081	GTO2	
082	RCLD	
083	F0?	calculate and print
084	1/X	termination resistance
085	RCL2	
086	×	
087	GSB6	
088	GTOb	

REGISTERS

0	1	2	3	4	5	6	7	8	9
$2\pi f_o$	Q	R		$\frac{1}{\omega_o R}$ or $\frac{R}{\omega_o}$	$\frac{R}{\omega_o}$ or $\frac{1}{\omega_o R}$	n	k	g_n	$A_{i,i+1}$
S0 g_1	S1 g_2	S2 g_3	S3 g_4	S4 g_5	S5 g_6	S6 g_7	S7 g_8	S8 g_9	S9 g_{10}

A	B	C	D	E	I
g_{11}	g_{12}	types 6 & 7 common element	R_T	g_{i+1}	index

2-5 **Program Listing II**

Step	Key	Comment
089	*LBL3	types 6 & 7 start
090	GSB9	initialize flags & regs
091	RCLi	recall and store g_n
092	STOE	
093	STO8	
094	RCL1	calculate and store $Q \cdot g_n$
095	×	
096	STOC	
097	*LBL4	types 6 & 7 loop start
098	F1?	if type 6 print common tank capacitor value
099	GSB5	
100	DSZI	increment indicies
101	EEX	
102	ST+7	
103	RCLi	recall g_i
104	RCLE	interchange g_i and g_{i+1}
105	X⇄Y	
106	STOE	
107	×	calculate $A_{i,\ i+1}$
108	√X	
109	1/X	
110	RCL9	interchange $A_{i,\ i+1}$ and $A_{i+1,\ i+2}$
111	X⇄Y	
112	STO9	
113	+	calculate type 6 tank L or type 7 tank Q
114	GSB0	
115	RCL9	calculate coupling element
116	RCL8	
117	×	
118	GSB8	
119	SPC	
120	RCL7	test for loop exit
121	RCL6	
122	X>Y?	
123	GTO4	
124	F1?	if type 6 print last tank capacitor
125	GSB5	
126	RCL9	calculate type 6 last tank L, or type 7 last tank Q
127	GSB0	
128	RCL2	print termination resistance
129	GSB6	
130	GTOb	
131	*LBL0	types 6 & 7 common element
132	RCL8	calculate and print type 6 tank inductor or type 7 tank capacitor
133	×	
134	CHS	
135	RCLC	
136	+	
137	GSB8	
138	F0?	if type 7 print common tank inductor value
139	GSB5	
140	GTOb	goto space and return

Step	Key	Comment
141	*LBL5	common element print subr
142	RCLC	
143	*LBL7	type 6 C or type 7 L subr
144	1/X	
145	RCL5	
146	×	
147	PRTX	
148	RTN	
149	*LBL8	type 6 L or type 7 Q subr
150	RCL4	
151	×	
152	PRTX	
153	RTN	
154	*LBL9	initialization subroutine
155	SPC	if flag 0 is set, clear flag 1 and vice-versa
156	SF1	
157	F0?	
158	CF1	
159	RCL0	if type 6: $\frac{R}{\omega_o} \to R_4$; $\frac{1}{R\omega_o} \to R_5$
160	1/X	
161	STO4	
162	STO5	
163	RCL2	if type 7: $\frac{1}{R\omega_o} \to R_4$; $\frac{R}{\omega_o} \to R_5$
164	F1?	
165	1/X	
166	ST÷4	
167	ST×5	
168	EEX	initialize k
169	STO7	
170	CLX	initialize $A_{n,\ n+1}$
171	STO9	
172	RCL6	calculate register index for g_n
173	9	
174	+	
175	STOI	
176	RCL2	recall termination R
177	*LBL6	print and space subroutine
178	PRTX	
179	*LBLb	space and return subroutine
180	SPC	
181	RTN	

LABELS					FLAGS
A load f_o	B load BW	C load R	D	E load type	0 type 1 or 7
a	b space & rtn	c	d	e	1 type 2 or 6
0 type 1 start	1 type 2 start	2 types 1 & 2 loop start	3 types 6 & 7 start	4 types 6 & 7 loop start	2 even branch
5 print common element	6 prt & space	7 C or L output	8 L or C output	9 initialize	3

SET STATUS				
FLAGS	ON	OFF	TRIG	DISP
0		■	DEG	FIX
1		■	GRAD	SCI
2		■	RAD	ENG
3				n ___

USERS CHOICE

<u>HP-67 suggested program changes.</u> The "print" mode of output can be changed to the "R/S" mode by changing like statements at line numbers 147, 152, and 178. The program execution will halt at each data output point and await restart by the user via the "R/S" key.

PROGRAM 2-6 NORMALIZED LOWPASS TO BANDPASS FILTER TRANSFORMATIONS, TYPES 8, 9, 10, AND 11.

Program Description and Equations Used

This program converts normalized lowpass filter element values to a set of four bandpass filter topologies [16], [56]. These four topologies are shown in Fig. 2-6.1 in normalized form (1 ohm, 1 radian/sec center frequency). The parameter A_{ij} is defined by Eq. (2-5.1). Types 8 and 9 are narrowband transformations of types 2 and 1, while types 10 and 11 are exact transformations of types 2 and 1 obtained by applying Norton transformations to the shunt elements of type 2 to form type 10, or to the series elements of type 1 to form type 11. This transformation process is detailed in the equation derivation section following Example 2-6.2. The types 8 and 9 narrowband transformations will only provide accurate results when the loaded Q (ratio of center frequency to bandwidth) is greater than 5 or so. This restriction is not present with types 10 or 11. Because the type 8 or 9 coupling element causes extra zeros of transmission at either dc or infinite frequency, the frequency response will be skewed away from the extra transmission zero frequencies as implied by Fig. 2-5.3. Figure 2-5.2 should be consulted for picking the filter type best suited to the center frequency, loaded Q, and impedance level of the intended application.

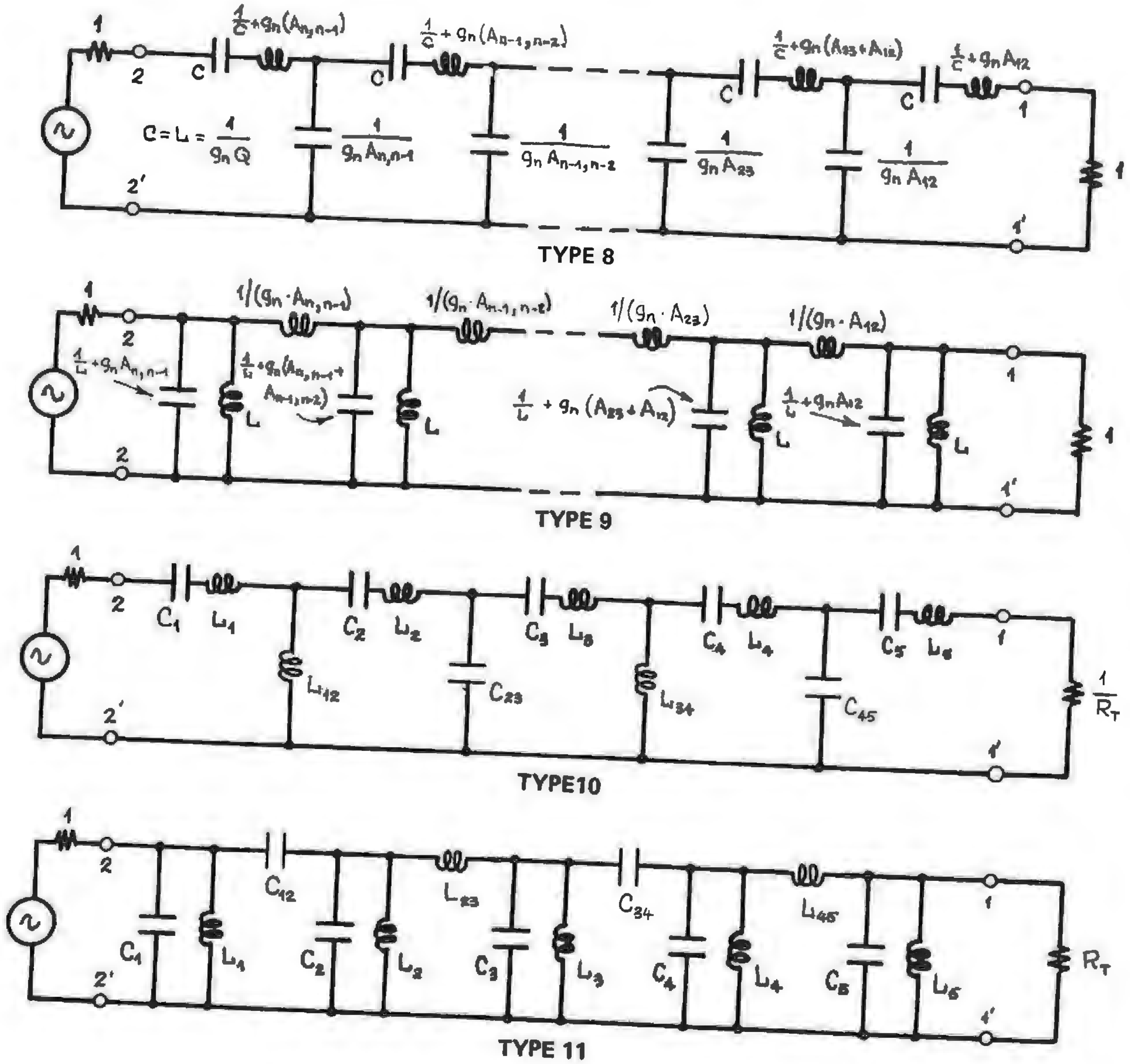


Figure 2-6.1 Normalized bandpass filter topologies for types 8, 9, 10, and 11.

Table 2-6.1 Types 10 and 11 normalized element values.

Type 10 element	Type 11 element	normalized element value
C_k	L_k	$\left(Q \cdot g_i - \frac{N_{i+2}-1}{Q \cdot g_{i+1}}\right)^{-1}$
L_k	C_k	$Q \cdot g_i - \frac{N_i - 1}{Q \cdot g_{i-1}}$
$L_{k,\,k+1}$	$C_{k,\,k+1}$	$\frac{N_i}{Q \cdot g_{i-1}}$
C_{k+1}	L_{k+1}	$\frac{1}{Q \cdot g_n}$
L_{k+1}	C_{k+1}	$Q \cdot g_n$
$C_{k+1,\,k+1}$	$L_{k+1,\,k+2}$	$\frac{Q \cdot g_{i-1}}{N_i}$.

$$N_i = \tfrac{1}{2}\left(1 + \sqrt{1 + 4Q^2 \cdot g_{i-1} \cdot g_n}\right) \quad ; \quad g_{n+1} \equiv 0$$

$k = 1, 3, 5, \ldots, n$ (n must be odd)

$i = n - k + 1$

The reverse ordering of the normalized lowpass coefficients from the element subscripts occurs because the dual form of the normalized lowpass filter is used. The dual is required to place the 1 ohm resistor next to the first shunt capacitor which is required for types 8 and 9 when transforming even ordered filters. Since the same register setup and recall routine is used for types 10 and 11, the dual form is carried over for convenience (it is not required).

Types 10 and 11 can be redrawn to show the ladder structure as T's or pi's of inductors and capacitors as shown in Fig. 2-6.2.

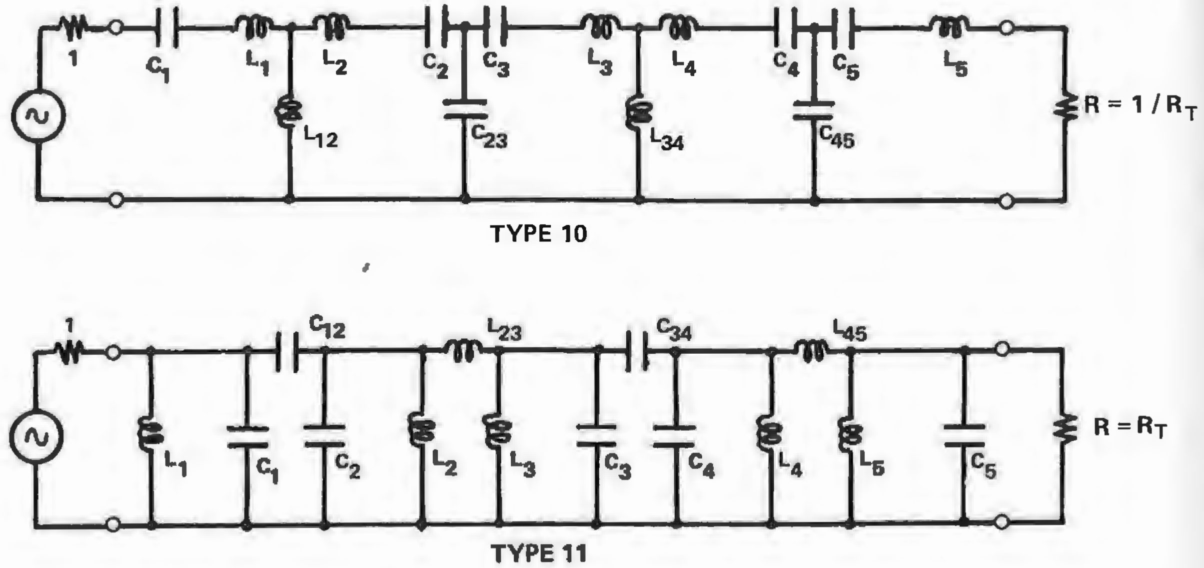


FIGURE 2-6.2 Types 10 and 11 showing T's and pi's of L's and C's.

These pi's and T's of inductors can be replaced with an active realization that contains only op-amps, resistors, and capacitors by using 2 back-to-back generalized impedance converter (GIC) circuits as detailed in Orchard and Sheahan's paper [42], and shown in Figs. 2-6.3 & 4.

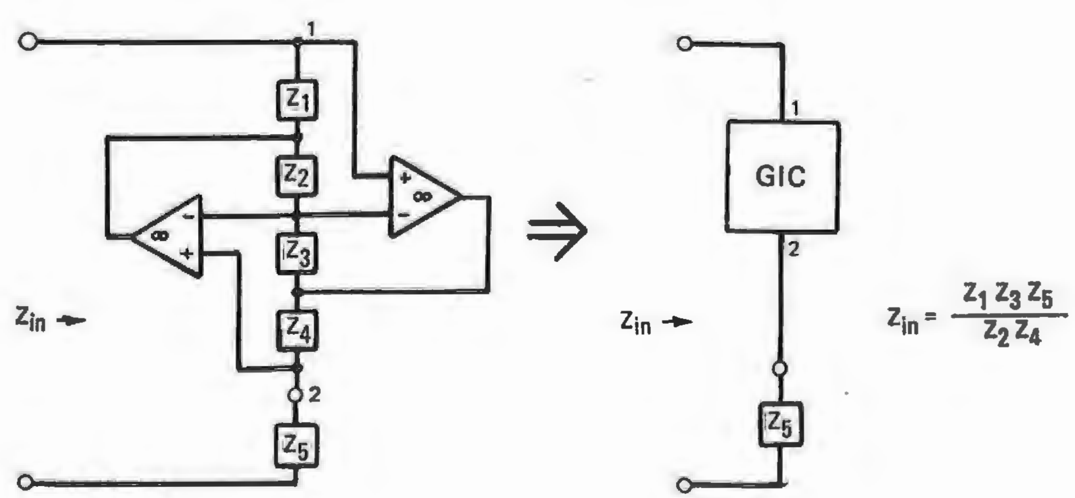


Figure 2-6.3 Antoniou GIC circuit [3].

If Z_1, Z_2, Z_3, and Z_5 are resistors and Z_4 is a capacitor, then,

$$Z_{in} = \frac{R_1 \; R_3 \; R_5}{R_2} \cdot sC_4 = sL \qquad (2\text{-}6.1)$$

Furthermore, if $R_2 = R_3$ (a Q enhancement condition), then,

$$L = R_1 C_4 R_5 \qquad (2\text{-}6.2)$$

Two GIC circuits with the component selection outlined above can be combined to produce a circuit that simulates a T or pi of inductances. These circuits are shown in Fig. 2-6.4.

Aside from the elimination of inductors, this particular mechanization is very easy to tune. Changing resistor R_1 in the GIC alters the apparent inductance seen at the terminals. The capacitor, C_4, needs to be stable (e.g., polystyrene or mica) but can have a large initial tolerance which is accommodated during the tuning procedure.

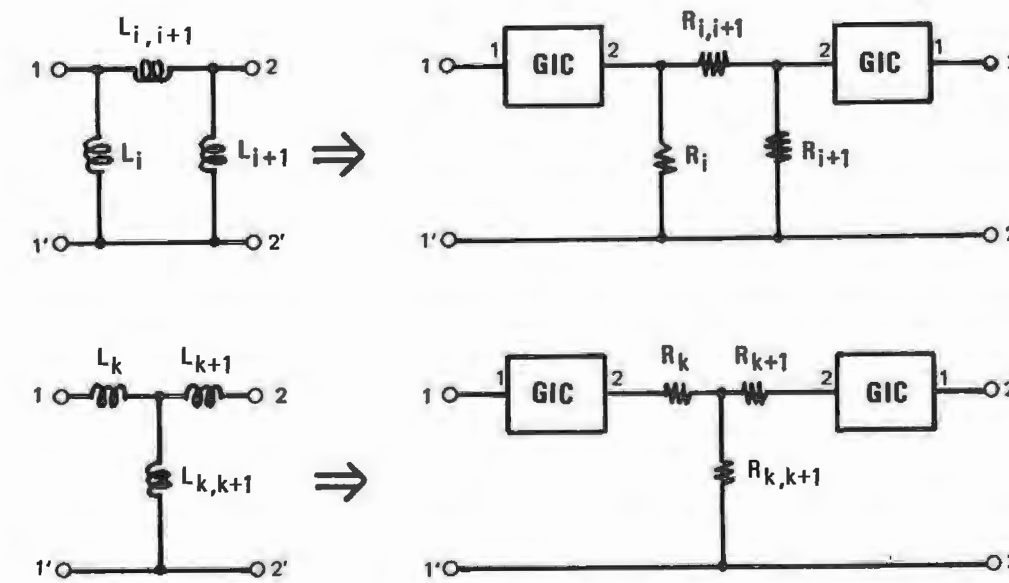


Figure 2-6.4 Pi or T inductance simulation circuits using GIC's.

The diagrams and discussion thus far have used the filters in normalized form, i.e., 1 ohm termination resistor, and 1 radian/second center frequency. The prototype filter is denormalized by multiplying each normalized inductor by $2\pi f_o/R$, and dividing each normalized capacitor by $2\pi f_o R$, where f_o and R are the desired center frequency and termination resistance level respectively. The program accomplishes this denormalization by calling either subroutine 7 or 8. For types 8 and 10, subroutine 7 denormalizes capacitors and subroutine 8 denormalizes inductors, and the reverse is true for types 9 and 11.

Tuning procedure for types 7, 9, and 11*. After the component values have been calculated, the inductors designed,** fabricated and adjusted to value, and the capacitors selected and padded to the proper value, the filter may be assembled and tuned so it will exhibit the desired response.

Tuning is accomplished by adjusting each of the parallel tank elements. For low frequency filters, the inductor is usually chosen as the adjustable element. At higher frequencies the capacitor is usually chosen as the adjustable element. The resonance of the tank circuit must include the effects of the coupling elements. By temporarily shorting out adjacent tank circuits, the coupling element influence will be included. This tuning procedure is described next.

1) Temporarily place a short at location "B" and adjust C_1 (or L_1) to resonate the tank circuit at the center frequency of the filter, f_o. The connection (short) must be low inductance with respect to the other inductances in the circuit.

2) Remove the short at "B," and temporarily place shorts at locations "A" and "C." Adjust C_2 (L_2) for tank circuit resonance at the filter center frequency.

3) Continue shorting out adjacent tanks with low inductance shorts at locations "B" & "D," "C" & "E," and "D," and adjusting each resulting tank circuit for resonance at the filter center frequency, f_o. These steps will complete the tuning of the filter.

** For more information on inductor design, see the ferromagnetic core and air core inductor design programs contained in another section of this book. Also see the Ferroxcube Inc. publication "Linear Ferrite Materials and Components" for information on ferrite pot core inductor design.

* See program 2-5 for the type 6, 8, and 10 tuning procedure.

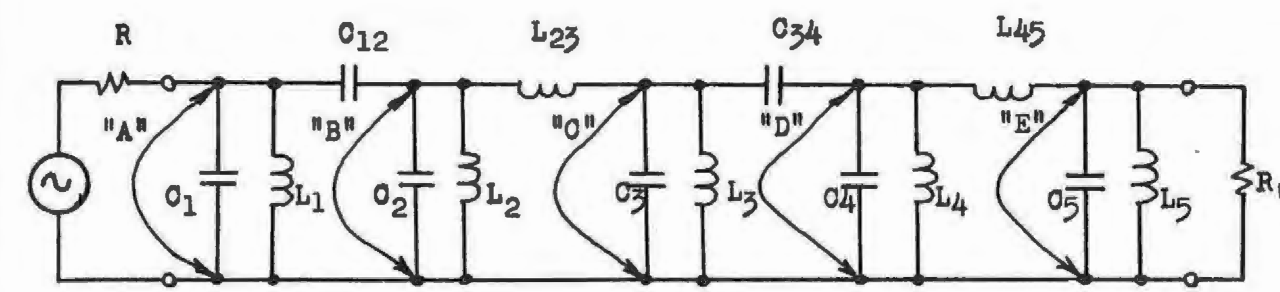


Figure 2-6.5 Circuit shorts for Types 7, 9, and 11 tuning.

2-6 # User Instructions

NORMALIZED LOWPASS TO BANDPASS TYPES 8, 9, 10, & 11				
load center frequency	load bandwidth	load termination resistance		load type # & start

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card (normalized lowpass coefficients and related parameters must be loaded into the registers either manually, or as output from Program 2-3)			
2	Load center frequency in Hz	f_o	A	ω_o
3	Load bandwidth in Hz	BW	B	Q
4	Load termination resistance in ohms	R	C	
5	Load type number of filter desired	8,9,10,11	E	load R
	The tank capacitor is always printed first independent of filter type.			C_{tank} L_{tank}
	*The coupling element will be as follows:			C,L cplg*
	Type 8, capacitor			
	Type 9, inductor			C_{tank}
	Type 10, alternating L's & C's			L_{tank}
	Type 11, alternating C's & L's			
	See Fig 2-6.1 for more details.			C,L cplg*
				C_{tank} ⋮
				load R

Example 2-6.1 Type 10 Filter Design

A Chebyshev response bandpass filter is required to pass a 20 Hz band of information geometrically centered about 1000 Hz with 0.5 dB ripple or less, and to operate in a 1000 ohm system. The filter stop-bandwidth is 60 Hz, and the filter should reject frequencies lying outside this band by at least 60 dB.

Referring to Fig. 2-5.2 under the headings f = 1 kHz, Q_L = 1000/20 = 50, and R = 500, type 8 is practical, and type 10 is readily realizable. Since active inductor simulation is anticipated, type 10 will be selected.

Program 2-1 is used to calculate the filter order necessary to meet the requirements, and Program 2-3 is used to calculate and store the normalized lowpass coefficients for use by this program. The HP-97 printout for all these programs is shown below.

Load Program 2-1 and calculate required filter order

	GSBe	select Chebyshev response
.50	GSBA	load Ap_{dB}
60.00	GSBB	load As_{dB}
3.00	GSBC	load λ and calculate minimum filter order
4.31	***	n, the minimum filter order (output)
5.00	GSBC	load integral filter order and calculate λ
2.31	***	λ to meet Ap_{dB} and As_{dB} given n = 5
3.00	GSBE	load λ required and calculate actual As_{dB}
61.40	***	As_{dB} for n = 5 and λ = 3

Load Program 2-3 and calculate Chebyshev LP normalized coefficients

5.	GSBA	load required filter order
1.	GSBB	load desired termination resistance ratio
.5	GSBC	load Chebyshev passband ripple and start
1.05926+00	***	normalized -3dB frequency (output)
1.00000+00	***	R_T (normalized port 1 termination resistance)
1.70577+00	***	g_1
1.22963+00	***	g_2
2.54083+00	***	g_3
1.22963+00	***	g_4
1.70577+00	***	g_5
1.00000+00	***	R (normalized port 2 termination resistance)

Example 2-6.1 (continued)

Load program 2-6 and calculate type 10 filter elements

1000.	GSBA	load center frequency
20.	GSBB	load bandwidth
1000.	GSBC	load termination resistor
10.	GSBE	select filter type and start
1.00000+03	***	R
1.86608-09	***	C1
13.3879+00	***	L1
188.752-03	***	L12
1.86608-09	***	C2
13.5741+00	***	L2
134.199-09	***	C23
1.26442-09	***	C_3
20.0331+00	***	L_3
188.752-03	***	L_{34}
1.86608-09	***	C_4
13.5741+00	***	L_4
134.199-09	***	C_{45}
1.89203-09	***	C_5
13.5741+00	***	L_5
1.00000+03	***	R_T

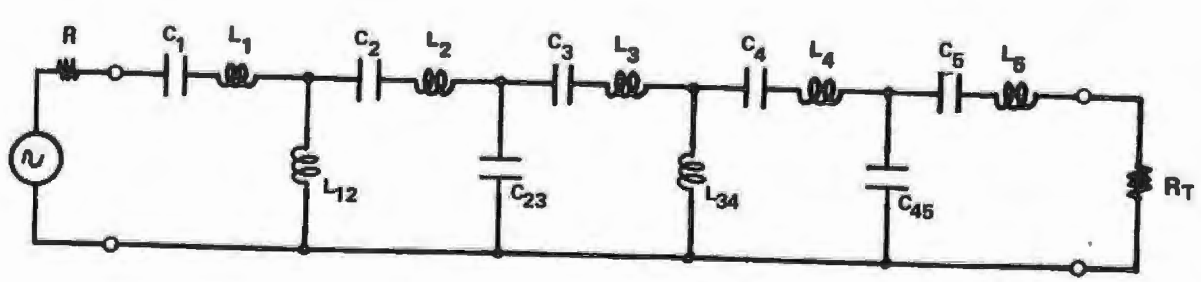


Figure 2-6.6 Type 10 bandpass filter schematic.

Example 2-6.2 Singly terminated type 10 filter design

Because the type 10 filter is an exact bandpass transformation of the lowpass prototype (as is the type 11), the terminating resistances need not be equal. This example will show the synthesis of a singly terminated type 10 filter, i.e., R_T is allowed to approach infinite resistance. The equally terminated filter case is the least sensitive to component value changes. When the filter is singly terminated, the operating Q's of the tank circuits become higher as the open (or shorted) end of the filter is approached. This means that the changes in tank Q's will have a greater effect on the overall operating Q of the tank in the filter, and hence, the filter response. The HP-97 printout for the singly terminated type 10 filter follows. Refer to Fig. 2-6.6 for the filter schematic.

Load Program 2-3

5.	GSBA	load n
1.+05	GSBB	load R_T ratio
.5	GSBD	load Chebyshev ripple
1.05926+00	***	ω_{-3dB} (output)
100.000+03	***	R_T
1.53866+00	***	g_1
1.64272+00	***	g_2
1.81407+00	***	g_3
1.42917+00	***	g_4
852.839-03	***	g_5
1.00000+00	***	R

Load Program 2-6

```
      1000. GSBA     load fo
        20. GSBB     load bandwidth
      1000. GSBC     load termination R
        10. GSBE     select type & start

 1.00000+03  ***     R

 3.73236-09  ***     C1
 6.66484+00  ***     L1

124.064-03  ***      L12

 3.73236-09  ***     C2
 6.78668+00  ***     L2

204.171-09  ***      C23

 1.76961-09  ***     C3
14.3222+00  ***      L3

115.645-03  ***      L34

 3.73236-09  ***     C4
 6.78668+00  ***     L4

219.026-09  ***      C45

 2.08814-09  ***     C5
12.2443+00  ***      L5

10.0000-03  ***      RT   (short circuit)
```

<u>Derivation of types 10 and 11 transformations</u>

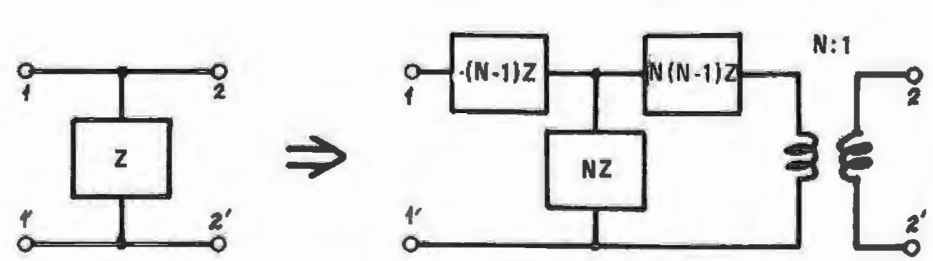


Figure 2-6.7 Norton's second transformation.

Figure 2-6.7 shows one form of Norton's second transformation [39]. This transformation changes a single shunt impedance into a T of impedances, one of which is negative, plus an ideal transformer with turns ratio N. Figure 2-6.8 shows how a parallel resonant tank circuit can be changed into a section of a type 10 bandpass filter structure.

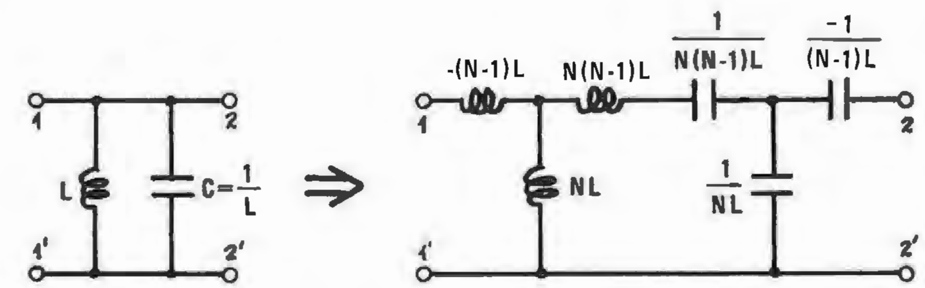


Figure 2-6.8 Norton's second transformation applied to a parallel LC tank circuit.

In Fig. 2-6.8, Norton's transformation has been applied back-to-back, i.e., the 2-2' terminals of the Norton transformation of the inductor have been connected to the 2-2' terminals of the Norton transformation of the capacitor. The same transformer ratio, N, is used for both transformations, therefore, the two ideal transformers are back-to-back providing an overall transformer ratio of unity and can be eliminated.

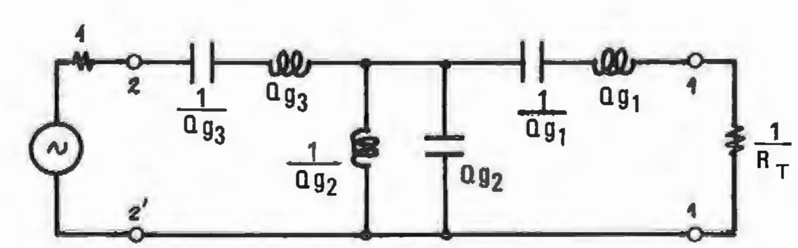


Figure 2-6.9 Type 2 normalized bandpass filter obtained from lowpass prototype (note port ordering).

Figure 2-6.9 shows a type 2 normalized bandpass filter obtained from the transformation of a lowpass prototype. The dual lowpass form is used (see Fig. 2-3.1 lower) and is scaled to a cutoff frequency of 1/Q (Q is the ratio of the filter center frequency to bandwidth); each frequency scaled series lowpass inductor is series resonated with a capacitor at $\omega = 1$, and each shunt scaled lowpass capacitor is parallel resonated with an inductor at $\omega = 1$. Next, the circuit of Fig. 2-6.8 is substituted for the parallel resonant tank, and the negative elements in the series arms combined with the positive series elements of Fig. 2-6.9. The results of this process yield the topology shown in Fig. 2-6.10. Higher ordered (odd order) filters are obtained by repeated application of this procedure.

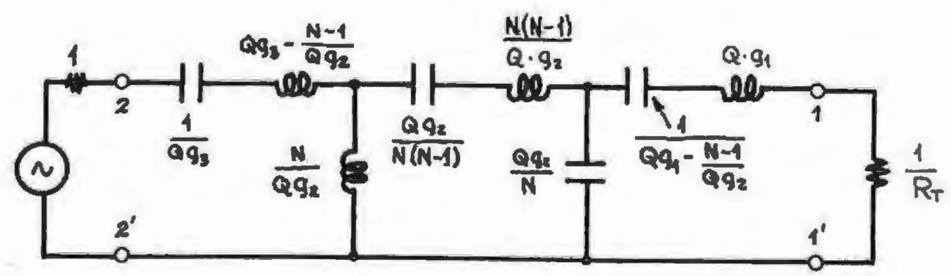


Figure 2-6.10 Type 10 normalized bandpass filter resulting from transformation.

The type 11 bandpass filter is shown in Fig. 2-6.1 and is the dual of the type 10 structure. Type 11 can be derived in a manner similar to the type 10 procedure by applying Norton's first transformation to a type 1 normalized bandpass filter. Norton's first transformation is shown in Fig. 2-8.1. Since type 11 is the dual of type 10 it can be more directly derived from the type 10 structure itself as shown by Fig. 2-6.1 and Table 2-6.1.

The value for N_i given in Table 2-6.1 is derived by making the transformed tank capacitor (inductor) value the same as the first ladder tank capacitor (inductor) for type 10 (11), i.e.,

$$\frac{1}{Q \cdot g_n} = \frac{Q \cdot g_i - 1}{N_i \cdot (N_i - 1)} \qquad (2\text{-}6.3)$$

Solving for N_i yields:

$$N_i = \tfrac{1}{2}\left(1 + \sqrt{1 + 4Q^2 \cdot g_{i-1} \cdot g_n}\right) \qquad (2\text{-}6.4)$$

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD CENTER FREQUENCY
002	Pi	
003	ENT↑	
004	+	form and store $2\pi f_o \rightarrow R0$
005	×	
006	STO0	
007	RTN	
008	*LBLB	LOAD FILTER BANDWIDTH
009	Pi	
010	ENT↑	
011	+	
012	×	form and store:
013	RCL0	
014	X⇄Y	$Q_L = \frac{2\pi f_o}{2\pi BW} \rightarrow R_1$
015	÷	
016	STO1	
017	RTN	
018	*LBLC	LOAD TERMINATION RESISTANCE
019	STO2	
020	RTN	
021	*LBLE	LOAD FILTER TYPE AND START
022	2	
023	÷	calculate starting label index
024	STOI	
025	INT	
026	4	
027	-	
028	X<0?	generate "ERROR" if filter type is less than 8
029	GTOa	
030	X⇄I	store label index
031	SF0	
032	FRC	set flag 0 if order is odd
033	X=0?	
034	CF0	
035	GTOi	goto starting label
036	*LBL0	type 8 and 9 routine
037	GSB9	initialize registers
038	RCLi	
039	STOE	recall and store g_1 for dual filter topology
040	STO8	
041	RCL1	
042	×	calculate and store common element value reciprocal
043	STOC	
044	CLX	
045	STO9	initialize $A_{01} = 0$
046	*LBL2	types 8 & 9 loop start
047	F1?	print type 9 tank capacitor
048	GSB5	
049	DSZI	
050	EEX	increment indices
051	ST+7	
052	RCLi	recall g_{i+1}
053	RCLE	
054	X⇄Y	recall g_i and store g_{i+1}
055	STOE	
056	×	
057	√X	calculate $A_{i,\ i+1}$
058	1/X	
059	RCL9	
060	X⇄Y	interchange $A_{i-1,i}$ & $A_{i,i+1}$
061	STO9	
062	+	form $A_{i-1,i} + A_{i,\ i+1}$, and
063	GSB0	output related element
064	RCL9	
065	RCL8	calculate and output coupling element
066	×	
067	GSB7	
068	SPC	
069	RCL7	
070	RCL6	test for loop exit
071	X>Y?	
072	GTO2	
073	F1?	output type 9 tank
074	GSB5	capacitor
075	RCL9	output rest of last
076	GSB0	tank circuit
077	RCL2	
078	GSB6	recall and print terminating
079	GTOb	resistance value
080	*LBL0	types 8 & 9 output routine
081	RCL8	
082	×	output type 8 tank
083	RCLC	capacitor, or type 9
084	+	tank inductor
085	GSB8	
086	F0?	output type 8 tank
087	GSB5	inductor
088	GTOb	
089	*LBL1	types 10 & 11 routine start
090	GSB9	initialize registers
091	*LBL3	types 10 & 11 loop start
092	RCL9	$Q \cdot g_{i+1}$
093	RCLi	
094	RCL1	$i = n, n-2, \cdots, 1$
095	×	$Q \cdot g_i$
096	STO9	
097	X⇄Y	
098	RCLE	N_{i+2} $(N_{n+2} \equiv 1)$
099	EEX	
100	-	
101	X⇄Y	
102	÷	
103	-	
104	STO3	$Q \cdot g_i - \frac{N_{i+2} - 1}{Q g_{i+1}}$
105	F3?	if first time through loop,
106	STOC	store value of first L or C
107	F1?	output type 11 tank
108	GSB7	capacitor
109	2	increment index, k
110	ST+7	

REGISTERS

0 $2\pi f_o$	1 Q_L	2 R	3 scratch	4 $\frac{1}{\omega_o R}$, $\frac{R}{\omega_o}$	5 $\frac{R}{\omega_o}$, $\frac{1}{\omega_o R}$	6 n	7 k	8 scratch	9 scratch
S0 g_1	S1 g_2	S2 g_3	S3 g_4	S4 g_5	S5 g_6	S6 g_7	S7 g_8	S8 g_9	S9 g_{10}
A g_{11}	B g_{12}	C common element	D R_T	E N , g_{i-1}	I index				

Program Listing II

Step	Key	Comment
111	RCL6	
112	RCL7	test for loop exit
113	X>Y?	
114	GTO0	
115	RCL9	calculate and store
116	DSZI	transformer ratio for
117	RCLi	Norton transformation:
118	RCL1	
119	×	
120	STO9	
121	4	$N_i = \frac{1}{2}\left(1 + \sqrt{1 + 4Q^2 g_n \cdot g_{i-1}}\right)$
122	×	
123	RCLC	
124	×	
125	EEX	
126	+	
127	√X	
128	EEX	
129	+	
130	2	
131	÷	
132	STOE	
133	EEX	calculate and print tank
134	-	inductor for type 10 or
135	RCL9	tank capacitor for type 11
136	÷	
137	-	
138	GSB8	
139	RCL3	print type 11 tank inductor
140	F0?	
141	GSB7	
142	SPC	
143	RCLE	
144	RCL9	calculate and print
145	÷	coupling element, L for
146	STO8	type 10 or C for type 11
147	GSB8	
148	SPC	print type 10
149	RCLC	tank capacitor
150	F1?	
151	GSB7	print type 10 tank
152	RCLC	inductor, or print type 11
153	GSB8	tank capacitor
154	RCLC	
155	F0?	print type 11
156	GSB7	tank inductor
157	SPC	
158	RCL8	calculate and print
159	GSB7	coupling element, C for
160	SPC	type 10 or L for type 11
161	DSZI	decrement index and
162	GTO3	return to loop start
163	*LBL0	last tank output:
164	RCL9	C for type 10, or
165	GSB8	L for type 11
166	RCL3	output type 10
167	F0?	tank inductor
168	GSB7	
169	SPC	calculate and print
170	RCL2	termination resistance
171	RCLD	
172	F1?	
173	1/X	
174	×	
175	GSB6	
176	GTOb	
177	*LBL5	common element output subr
178	RCLC	recall common element
179	*LBL7	L/C (odd/even) output subr
180	1/X	
181	RCL5	
182	×	
183	PRTX	
184	RTN	
185	*LBL8	C/L (odd/even) output subr
186	RCL4	
187	×	
188	PRTX	
189	RTN	
190	*LBL9	initialization subroutine
191	SPC	
192	SF1	
193	F0?	flag 1 $\overline{\text{flag 0}}$
194	CF1	
195	SF3	
196	RCL0	
197	1/X	setup denormalization
198	STO4	constants for L's and C's
199	STO5	(register order changed
200	RCL2	depending upon filter type
201	F1?	being odd or even)
202	1/X	
203	ST÷4	
204	ST×5	
205	EEX	
206	STO7	initialize registers
207	STO9	
208	STOE	
209	RCL6	
210	9	initialize normalized LP
211	+	coef recall index register
212	STOI	
213	RCL2	recall termination R
214	*LBL6	print and space subroutine
215	PRTX	
216	*LBLb	space and return subroutine
217	SPC	
218	RTN	

LABELS

A load f_o	B Load BW	C load R	D	E Load type
a	b space & rtn	c	d	e
0 types 8&9 start	1 types 10&11 start	2 types 8&9 loop start	3 types 10&11 loop start	4
5 print common elt.	6 print, spc, return	7 print C or L	8 print L or C	9 initialize

FLAGS

0	type 9 or 11
1	type 8 or 10
2	
3	first time thru loop

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	DEG	FIX
1		■	GRAD	SCI
2			RAD	ENG ■
3		■		n 5

HP-67 suggested program changes. To change from the "print" to "R/S" mode for program output, make the respective change at the following line numbers: 183, 188, and 217. The program will now stop at output points and await restart via the "R/S" command from the keyboard.

PROGRAM 2-7 WYE-DELTA TRANSFORMATIONS FOR R, L, OR C.

Program Description and Equations Used

This program performs the Y-Δ transformation for groups of three resistors, capacitors, or inductors. These transformations find use whenever awkward or physically impractical element values result from electrical network design. The resistive transformation is often used with operational amplifier summing network design to keep the resistor values low. The inductive and capacitive transformations can be of assistance in filter design.

The Y-Δ transformations for one-of-a-kind elements are summarized below:

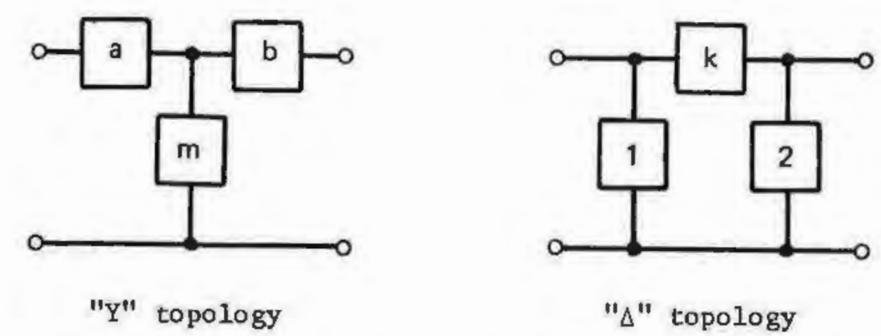


Figure 2-7.1 "Y" and "Δ" topology definitions.

For capacitors as network elements:

Y→Δ

$C_1 = C_a \cdot C_m / \Sigma C$

$C_2 = C_b \cdot C_m / \Sigma C$

$C_k = C_a \cdot C_b / \Sigma C$

$\Sigma C = C_a + C_b + C_m$

Δ→Y

$C_a = \Sigma CC / C_2$

$C_b = \Sigma CC / C_1$

$C_m = \Sigma CC / C_k$

$\Sigma CC = C_1 C_2 + C_2 C_k + C_1 C_k$

For inductors or resistors as network elements (read L's as R's):

Y→Δ

$$L_1 = \Sigma LL/L_b$$
$$L_2 = \Sigma LL/L_a$$
$$L_k = \Sigma LL/L_m$$
$$\Sigma LL = L_aL_b + L_aL_m + L_bL_m$$

Δ→Y

$$L_a = L_1 \cdot L_k/\Sigma L$$
$$L_b = L_2 \cdot L_k/\Sigma L$$
$$L_m = L_1 \cdot L_2/\Sigma L$$
$$\Sigma L = L_1 + L_2 + L_k$$

User Instructions

Y→Δ, Δ→Y TRANSFORMATION FOR L, R, OR C

set C	set L or R			print elements
load elt 1 or a	load elt k or m	load elt 2 or b	Y→Δ	Δ→Y

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load program card (one sided card)			
2	Select element type:			
	if capacitors		f A	
	if inductors, or resistors		f B	
3	Load element values			
	Load element 1 or a		A	
	Load element k or m		B	
	Load element 2 or b		C	
4	Select transformation type:			
	Y→Δ transformation		D	element a
				element m
				element b
				Σ a,m,b
				element 1
				element k
				element 2
				Σ 1,k,2
	Δ→Y transformation		E	element 1
				element k
				element 2
				Σ 1,k,2
				element a
				element m
				element b
				Σ a,m,b
5	To print presently stored elements		f E	elt 1,a
				elt k,m
				elt 2,b
				Σ elts.

Example 2-7.1

Convert the Y network of Fig. 2-7.2 into an equivalent Δ network. Compute the total capacitance both before and after the transformation.

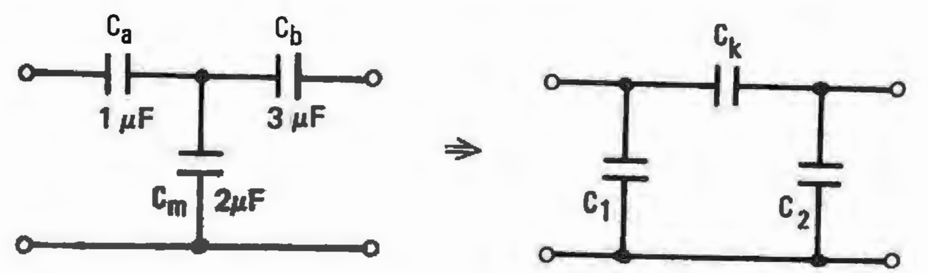


Figure 2-7.2 Capacitor networks for Example 2-7.1.

HP-97 printout

```
1.-06 GSB[illegible]   Ca  } load capacitor values
2.-06 GSBE             Cm  }
3.-06 GSBC             Cb  }
      GSBc             select capacitors
      GSBD             perform Y→Δ  transformation

1.000-06   ***   Ca    }
2.000-06   ***   Cm    } before transformation
3.000-06   ***   Cb    }
6.000-06   ***   ΣC's  }

333.3-09   ***   C1    }
500.0-09   ***   Ck    } after transformation
1.000-06   ***   C2    }
1.833-06   ***   ΣC's  }
```

As a result of the transformation, the total capacity has been reduced by 69.4%.

Example 2-7.2

A top coupled parallel resonant bandpass filter of the type 7 topology has been designed with the element values shown in Fig. 2-7.3. The 1 picofarad coupling capacitor is a problem since it is the same relative value as the parasitic (stray) capacities of the printed circuit board. By converting from a Δ capacitor configuration to a Y configuration, the minimum filter capacity is 202 pF as seen in Fig. 2-7.4, and the parasitic capacities of the printed circuit board are easily managed.

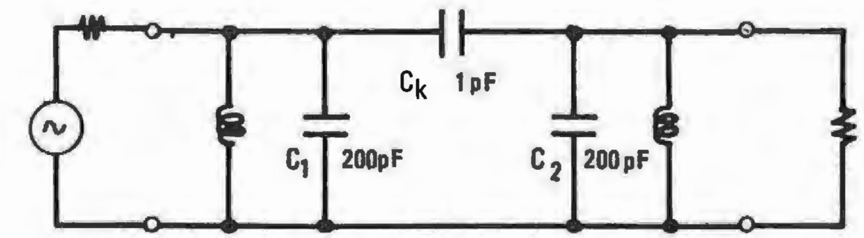
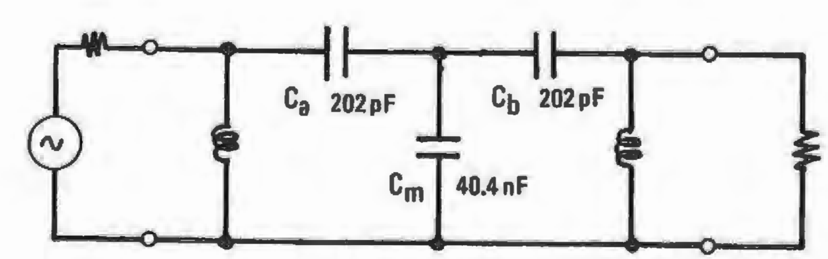


Figure 2-7.3 Type 7 filter design.

HP-97 printout for Δ→Y transformation:

```
200.-12 GSB[illegible]   C1  }
        GSBC             C2  } load capacitor values
   .-12 GSBE             Ck  }
        GSBa             select capacitors
        GSBE             start Δ→Y transformation

200.0-12   ***   C1              }
1.000-12   ***   Ck              } summary before
200.0-12   ***   C2              } transformation
401.0-12   ***   total capacity  }

202.0-12   ***   Ca              }
40.40-09   ***   Cm              } summary after
202.0-12   ***   Cb              } transformation
40.80-09   ***   total capacity  }
```

Figure 2-7.4 Network after Δ→Y transformation.



Program Listing

```
*LBLA   LOAD element 1 or a
 STOA
  RTN
-------------------------------
*LBLB   LOAD element k or m
 STOB
  RTN
-------------------------------
*LBLC   LOAD element 2 or b
 STOC
  RTN
-------------------------------
*LBLe   PRINT ELEMENTS
  SPC
 RCLA
 PRTX
 RCLB
 PRTX
    +
 RCLC
 PRTX
    +
 PRTX
  SPC
  RTN
-------------------------------
*LBLD   START Y→Δ TRANSFORMATION
 GSBe   print elements
 F0?    jump if L or R
 GTO0
*LBL1   Δ→Y for L or R, Y→Δ for C
 RCLA   form and store ΣX where
 RCLB   X is L, R, or C
    +
 RCLC
    +
 STOD
 RCLA   calculate element a or 1
 RCLB   and store in scratchpad
    ×
 RCLD
    ÷
 STOE
 RCLA   calculate element m or k
 RCLC
    ×
 RCLD
    ÷
 RCLE   store element a or 1
 STOA
   R↓
 STOE   temporarily store
        element m or k in scratchpad
 RCLB   calculate element b or 2
 RCLC
    ×
 RCLD
    ÷
 STOC
 RCLE   store element m or k
 STOB
 GTOe   print element values
-------------------------------
*LBLE   START Δ→Y TRANSFORMATION
 GSBe   print elements
 F0?    jump if L or R
 GTO1
*LBL0   Y→Δ for L or R, Δ→Y for C
 RCLA   form and store ΣXX where
 RCLB   X is L, R, or C
    ×
 RCLB
 RCLC
    ×
    +
 RCLA
 RCLC
    ×
    +
 STOD
 RCLC   calculate element 2 or c
    ÷   and store in scratchpad
 STOE
 RCLD   calculate element k or m
 RCLB
    ÷
 STOB
 RCLD   calculate element 1 or a
 RCLA
    ÷
 STOC
 RCLE   store element 2 or c
 STOA
 GTOe   print element values
-------------------------------
*LBLa   SET CAPACITORS AS ELEMENTS
  CF0
  RTN
-------------------------------
*LBLb   SET INDUCTORS OR RESISTORS
  SF0   AS ELEMENTS
  RTN
```

REGISTERS

0	1	2	3	4	5	6	7	8	9
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A element 1 or a	B element k or m	C element 2 or b	D ΣX or ΣXX X is L, R, or C	E scratchpad	I

LABELS

A load 1 or a	B load k or m	C load 2 or b	D Y→Δ	E Δ→Y
a set C	b set L, R	c	d	e print elements
0 L or R dest	1 L or R dest	2	3	4
5	6	7	8	9

FLAGS

0	L or R
1	
2	
3	

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	USER'S CHOICE	
1			DEG	FIX
2			GRAD	SCI
3			RAD	ENG
				n___

PROGRAM 2-8 NORTON TRANSFORMATIONS.

Program Description and Equations Used

Two network equivalence transformations developed by Edward L. Norton are shown below. They can be extremely useful for modifying network element values or topology.

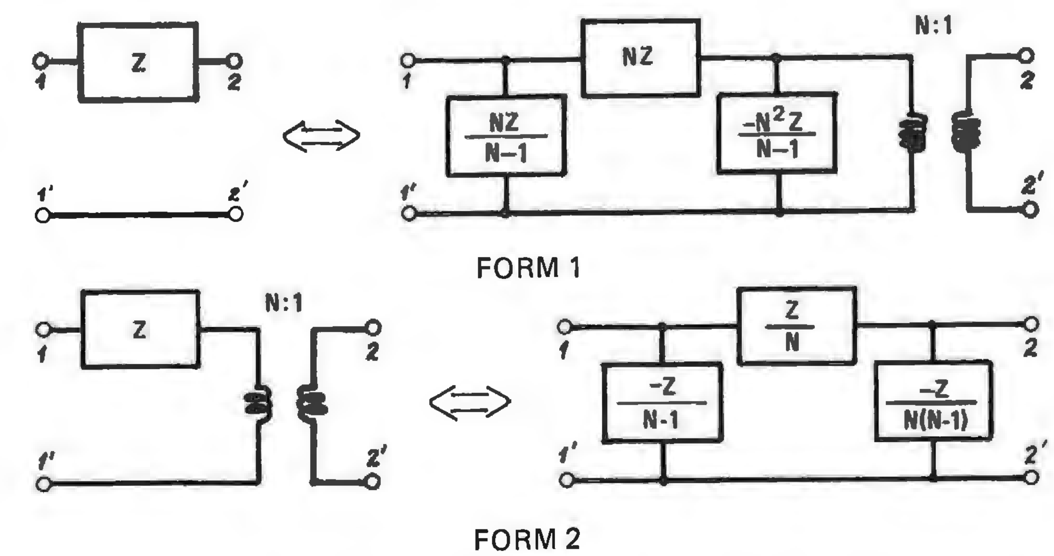


Figure 2-8.1 Two forms of Norton's first transformation.

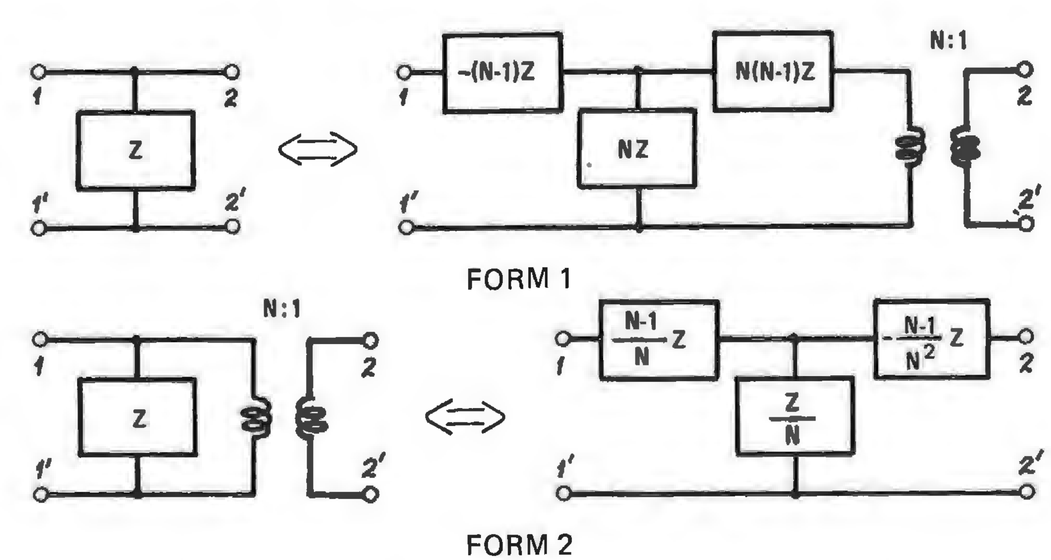
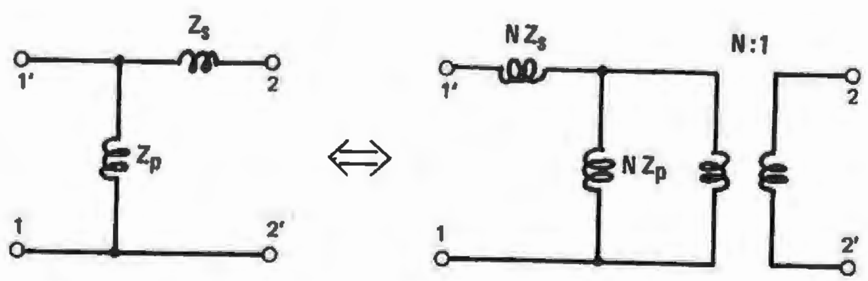


Figure 2-8.2 Two forms of Norton's second transformation.

Figure 2-8.1 shows two forms of Norton's first transformation, and Fig. 2-8.2 shows two forms of Norton's second transformation. The transformed network always contains a negative element, which is combined with a positive element not involved in the transformation. N must be chosen so this combination results in a zero or positive element value if the element is to be realized passively (there are active circuits which can simulate negative elements). When N is chosen so the negative and positive elements annihilate one-another, the overall network topology changes. This technique can be used to reverse an "L" network as shown in Fig. 2-8.3



$$N = \frac{Z_p}{Z_s + Z_p} < 1$$

Figure 2-8.3 Norton transformation applied to an "L" network.

Chapter 10 of Zverev [58] has many example of the application of Norton's transformations. Some insight into the power of Norton's transformations is related in the article "Reminiscences" by W.R. Bennett in CAS-24 no. 12 (Dec. 1977). Dr. Bennett recollects that Ed Norton could efficiently furnish a network to give a prescribed loss characteristic with the minimum number of elements by using only a very ordinary sliderule, his intuition, and his transformations.

This HP-67/97 program will transform either capacitors or inductors and resistors. Because the impedance of a capacitor is inversely proportional to the capacitance, multiplying an impedance by N has the effect of dividing the capacitance by N. Figure 2-8.4 shows form 1 of Norton's first transformation when the element being transformed is a capacitor.

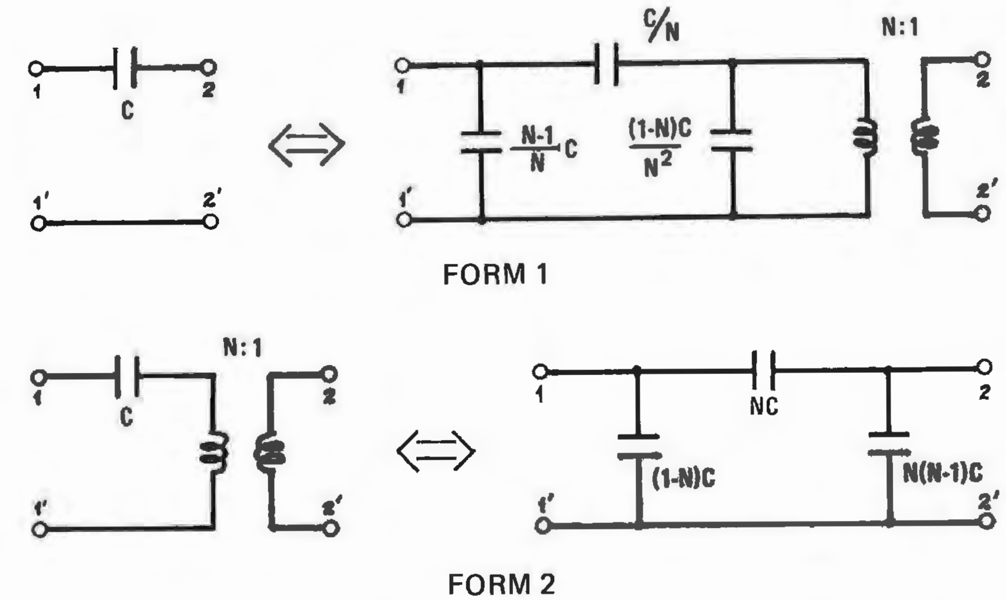


Figure 2-8.4 Norton's first transformation for capacitors.

The same reciprocal relations hold for Norton's second transformation as applied to capacitor networks.

User Instructions

NORTON TRANSFORMATIONS					
load C	OR	load L or R	load N	calculate 1st xfm forms	calculate 2nd xfm forms

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load program card			
2	If a capacitor network is being transformed, load capacitor value	C	A	
	OR			
	If an inductor or resistor network is being transformed, load L or R value	L, R	B	
3	Load ideal transformer ratio desired	N	C	
4	To calculate both forms of Norton's first transformation		D	
			form one	shunt elt
				series elt
				shunt elt
				space
				xfmr ratio
				space
				space
			form two	shunt elt
				series elt
				shunt elt
5	To calculate both forms of Norton's second transformation		E	
			form one	series elt
				shunt elt
				series elt
				space
				xfmr ratio
				space
				space
			form two	series elt
				shunt elt
				series elt

Example 2-8.1

An impedance stepdown of 3:1 is required at the output of the bandpass filter shown in Fig. 2-8.5. A transformer could be used to provide this function. Instead, use Norton's first transformation to provide the impedance stepdown without a transformer.

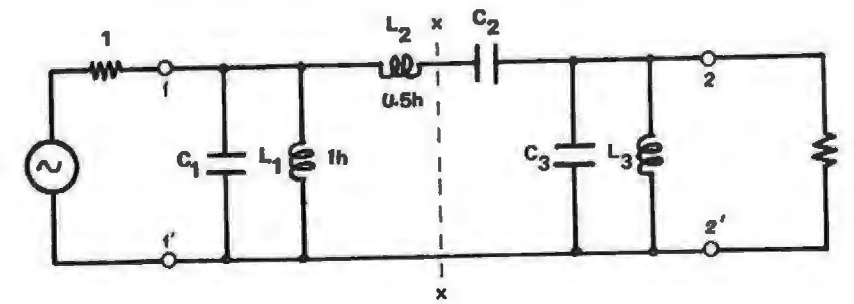


Figure 2-8.5 Bandpass filter network for Ex. 2-8.1.

A hypothetical $\sqrt{3}$:1 turns ratio transformer is inserted at x-x, and all network elements to the right scaled down in impedance by a factor of 3 as shown in Fig. 2-8.6.

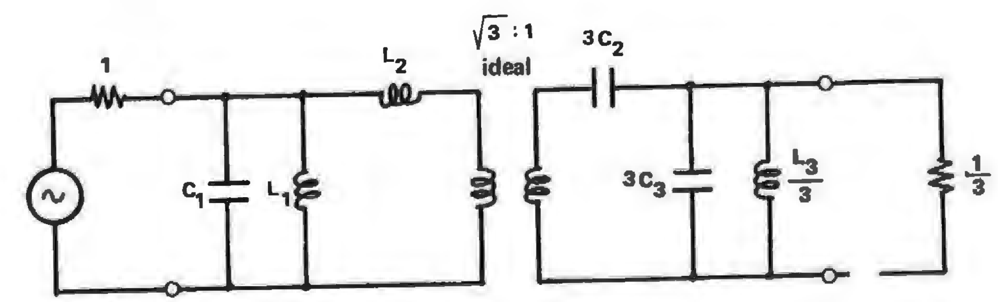


Figure 2-8.6 Network of Fig. 2-8.5 after insertion of hypothetical transformer.

Form 2 of Norton's first transformation is applied to L_2 and the transformer as shown in Fig. 2-8.7. The resulting negative shunt inductor is combined with L_1 as shown in Fig. 2-8.8.

HP-97 printout for Norton's first transformation

```
              .5 GSBE   L2

              3.     fX } N
                   GSBC }

                   GSBD   calculate Norton's first transformation

       1.183+00    ***    La               }
       866.0-03    ***    Lb               }
      -2.045+00    ***    Lc               }  form 1
                                           }
       1.732+00    ***    transformer ratio}

      -683.0-03    ***    La }
       288.7-03    ***    Lb }  form 2
       394.3-03    ***    Lc }
```

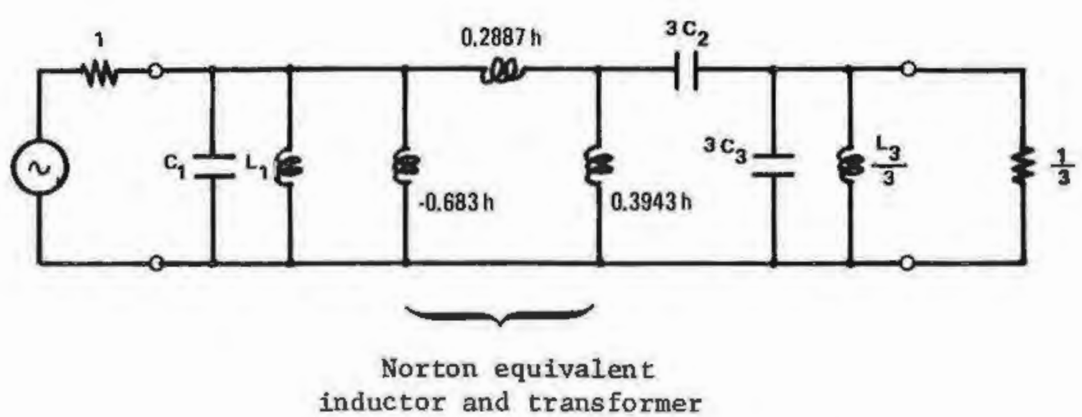


Figure 2-8.7 Network of Fig. 2-8.6 with form 2 of Norton's first transformation applied.

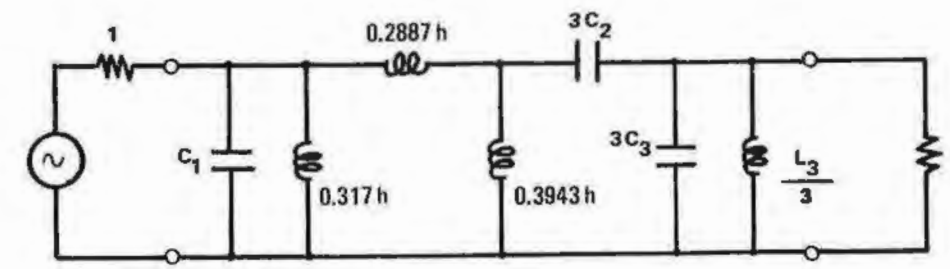


Figure 2-8.8 Final network with all negative elements absorbed.

Program Listing I

Step	Key	Comments
001	*LBLA	LOAD C
002	SF0	indicate capacitor entry
003	1/X	form and store reciprocal
004	STO0	of entry
005	1/X	restore entry
006	RTN	
007	*LBLB	LOAD L OR R
008	CF0	indicate L or R entry
009	STO0	store entry
010	RTN	
011	*LBLC	LOAD N
012	STO1	
013	RTN	
014	*LBLD	CALCULATE FIRST TRANSFORM
015	SPC	
016	RCL1	
017	RCL1	form 1 shunt element
018	EEX	calculation
019	-	
020	÷	
021	RCL0	
022	x	
023	STO2	
024	GSB0	
025	RCL0	
026	RCL1	form 1 series element
027	x	calculation
028	GSB0	
029	RCL2	
030	RCL1	form 1 shunt element
031	x	calculation
032	CHS	
033	GSB0	
034	SPC	
035	RCL1	recall and print transformer
036	PRTX	turns ratio
037	SPC	
038	SPC	
039	RCL0	
040	RCL1	form 2 shunt element
041	EEX	calculation
042	-	
043	÷	
044	STO2	
045	CHS	
046	GSB0	
047	RCL0	
048	RCL1	form 2 series element
049	÷	calculation
050	GSB0	
051	RCL2	form 2 shunt element
052	GTO1	calculation
053	*LBLE	CALCULATE SECOND TRANSFORM
054	SPC	
055	RCL0	
056	RCL1	form 1 series element
057	EEX	calculation
058	-	
059	x	
060	STO2	
061	CHS	
062	GSB0	
063	RCL0	
064	RCL1	form 1 shunt element
065	x	calculation
066	GSB0	
067	RCL2	
068	RCL1	form 1 series element
069	x	calculation
070	GSB0	
071	SPC	
072	RCL1	recall and print ideal
073	PRTX	transformer turns ratio
074	SPC	
075	SPC	
076	RCL1	
077	EEX	form 2 series element
078	-	calculation
079	RCL1	
080	÷	
081	RCL0	
082	x	
083	STO2	
084	GSB0	
085	RCL0	
086	RCL1	form 2 shunt element
087	÷	calculation
088	GSB0	
089	RCL2	
090	CHS	
091	*LBL1	
092	RCL1	form 2 series element
093	÷	calculation
094	GSB0	
095	SPC	
096	SPC	
097	RTN	
098	*LBL0	output subroutine
099	F0?	
100	1/X	
101	PRTX	
102	RTN	

REGISTERS

0	1	2	3	4	5	6	7	8	9
1/C or L	N	scratch							
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
A	B	C	D	E	I				

Program Listing II

LABELS					FLAGS
A load C	B load L	C load N	D calc type 1	E calc type 2	0 capacitor
a	b	c	d	e	1
0 output subr.	1	2	3	4	2
5	6	7	8	9	3

SET STATUS

FLAGS			TRIG	DISP
	ON	OFF	USER'S CHOICE	
0	■		DEG	FIX
1			GRAD	SCI
2			RAD	ENG
3				n____

PROGRAM 2-9 BUTTERWORTH AND CHEBYSHEV ACTIVE LOWPASS FILTER DESIGN AND POLE LOCATIONS.

Program Description and Equations Used

This program calculates the pole locations and Sallen and Key topology element values for un-normalized Butterworth or Chebyshev all pole lowpass filter approximations.

The program is designed to allow the use of capacitors with specified values as might result from the actual capacity measurement of a selected capacitor. The design process starts by assuming that all resistors are equal to the design resistance level, and the capacitor values are calculated to meet the filter requirements. The user may select new capacitor values near the original values, and the program will calculate new resistor values to meet the filter requirements. These resistor values can generally be selected from the nearest standard 0.1% resistor values.

The normalized pole locations of a Butterworth lowpass filter lie on a circle of unit radius as shown by Fig. 2-2.1 with the generalized pole locations given by Eqs. (2-2.12) and (2-2.13). The normalized pole locations for a Chebyshev lowpass filter lie on an ellipse as shown by Fig. 2-2.3 with the generalized pole locations given by Eqs. (2-2.15), (2-2.16), (2-2.17), and (2-2.18).

Each complex conjugate pole pair can be expressed in either the cartesian (real and imaginary parts) or the polar (magnitude and angle) co-ordinate systems. A variation on the polar system allows the pole pair to be defined in terms of the natural frequency (polar radius), ω_n, and "Q," or quality factor. The relationship between these co-ordinate systems is shown in Fig. 2-9.1, and described by Eqs. (2-9.1) through (2-9.3).

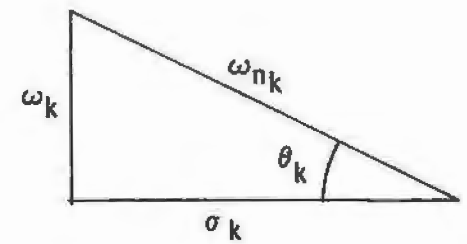


Figure 2-9.1 Co-ordinate system relationships.

$$\omega_{n_k} = \sigma_k^{\,2} + \omega_k^{\,2} \tag{2-9.1}$$

$$\theta_k = \tan^{-1}\left(\frac{\omega_k}{\alpha_k}\right) \tag{2-9.2}$$

$$Q_k = \frac{1}{2\cos\theta_k} = \frac{\omega_{n_k}}{2\alpha_k} \tag{2-9.3}$$

The element values for the Sallen and Key type active resonator are easily expressed in terms of ω_n and Q as follows:

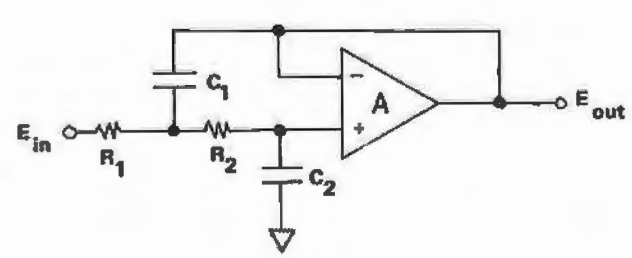


Figure 2-9.2 Sallen and Key active lowpass filter topology.

$$C_1 = \frac{Q(R_1 + R_2)}{\omega_n R_1 R_2} \qquad = \frac{2Q}{\omega_n R} \tag{2-9.4}$$

$$C_2 = \frac{1}{\omega_n Q (R_1 + R_2)} \qquad = \frac{C_1}{4Q^2} \tag{2-9.5}$$

$$R_1 = R_2 = R$$

The Sallen and Key resonator topology is chosen over other types because of the low parameter sensitivities to element changes. This type of filter synthesis is called the cascade method. Each pole pair is synthesized by an isolated op-amp resonator circuit. The entire filter is formed from a cascade of these resonator circuits. With each pole pair being independent, the overall filter sensitivities to component value changes are higher than an equivalent LC filter. See reference [49] (page 314) for more details.

If higher order filters are required (n greater than 9 or so), either the leapfrog (Szentirmai) topology using Deliyannis resonators [48], [20] or Cauer-Chebyshev filters using biquadratic sections [35] should be considered.

If the two capacitors in the Sallen and Key circuit are specified, then the following equations express the resistor values.

$$R_1 = \frac{1 + \sqrt{1 - 4Q^2 C_2/C_1}}{2Q\omega_n C_2} \tag{2-9.6}$$

$$R_2 = \frac{1}{\omega_n^{\,2} C_1 C_2 R_1} \tag{2-9.7}$$

To ensure the quantity under the radical is positive in the equation for R_1, C_2 should be selected to be a lower value, and C_1 a higher value than given by Eqs. (2-9.4) and (2-9.5).

If the filter order is odd, then a real pole exists. A third order op-amp resonator circuit may be used to produce both the real pole and a complex conjugate pair. The lowest Q pole pair is selected for realization by this circuit to minimize sensitivities, and to keep the element value spread within bounds. The third order section topology is shown in Fig. 2-9.3.

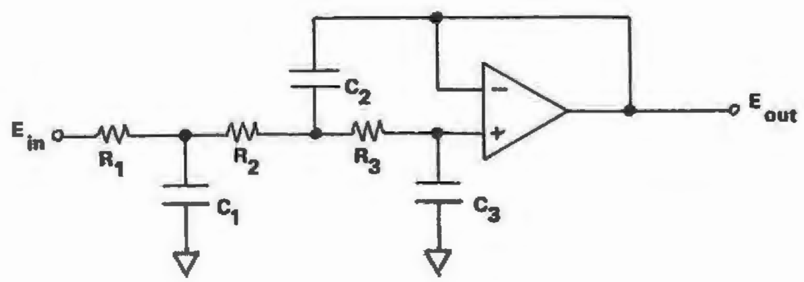


Figure 2-9.3 Third order op-amp resonator circuit.

$$\frac{E_{out}}{E_{in}} = \frac{1}{D(s)} \qquad (2\text{-}9.8)$$

where

$$D(s) = s^3C_1C_2C_3R_1R_2R_3 + s^2C_3\{C_1R_1(R_2 + R_3) + C_2R_3(R_1 + R_2)\} + s\{C_1R_1 + C_3(R_1 + R_2 + R_3)\} + 1$$

$$\frac{E_{out}}{E_{in}} = \frac{1}{Cs^3 + Bs^2 + As + 1} = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{s}{\omega_n Q} + 1} \cdot \frac{1}{\tau s + 1} \qquad (2\text{-}9.9)$$

$$\text{for } R_1 = R_2 = R_3 = 1 \qquad (2\text{-}9.10)$$

$$A = C_1 + 3C_3 = \tau + \frac{1}{\omega_n Q} \qquad (2\text{-}9.11)$$

$$B = 2C_3(C_1 + C_2) = \frac{\tau}{\omega_n Q} + \frac{1}{\omega_n^2} \qquad (2\text{-}9.12)$$

$$C_1 = C_1C_2C_3 = \frac{\tau}{\omega_n^2} \qquad (2\text{-}9.13)$$

The equations for A, B, and C represent three equations in three unknowns, C_1, C_2, and C_3. By algebraic manipulation, a cubic equation in C_1 alone may be obtained.

$$C_1^3 - C_1^2(A) + C_1\left(\frac{3}{2}B\right) - 3C = 0 \qquad (2\text{-}9.14)$$

A Newton-Raphson iterative solution is used to find the real root of this equation (there will be at least one). Once C_1 is found, the remaining two capacitors are found as follows:

$$C_3 = \frac{A - C_1}{3} \qquad (2\text{-}9.15)$$

$$C_2 = \frac{C}{C_1C_3} \qquad (2\text{-}9.16)$$

If the three capacitors are specified, then the transmission function (Eq. (2-9.9)) may be used to obtain three equations in terms of the three unknown resistors. Equating like powers of s, as before, these equations result:

$$A = C_1R_1 + C_3(R_1 + R_2 + R_3) = \tau + \frac{1}{\omega_n Q} \qquad (2\text{-}9.17)$$

$$B = C_3\{C_1R_1(R_2 + R_3) + C_2R_3(R_1 + R_2)\} = \frac{\tau}{\omega_n Q} + \frac{1}{\omega_n^2} \qquad (2\text{-}9.18)$$

$$C = C_1C_2C_3R_1R_2R_3 = \frac{\tau}{\omega_n^2} \qquad (2\text{-}9.19)$$

By algebraic manipulation, R_2 may be eliminated leaving two equations in two unknowns, R_3 as a cubic function of R_1 alone, and a quadratic equation in R_1 with R_3 as a parameter. The quadratic formula is used to reduce the second equation to R as a function of R_1 alone. These two non-linear equations in two unknowns are solved using an iterative method given in an unpublished paper by Robert Esperti of Delco Electronics.

$$R_3 = \frac{1}{R_1^2C_2C_3}\left\{R_1^2(C_1(C_1 + C_3)) + R_1^2(AC_1) + R_1(B) + \frac{C}{C_1}\right\} \qquad (2\text{-}9.20)$$

$$R_1 = \frac{-b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \qquad (2\text{-}9.21)$$

$$\frac{-b}{2a} = \frac{A - C_3R_3}{2(C_1 + C_3)} \; ; \quad \frac{c}{a} = \frac{C}{(C_1 + C_3)\cdot C_1C_2R_3} \qquad (2\text{-}9\text{-}22)$$

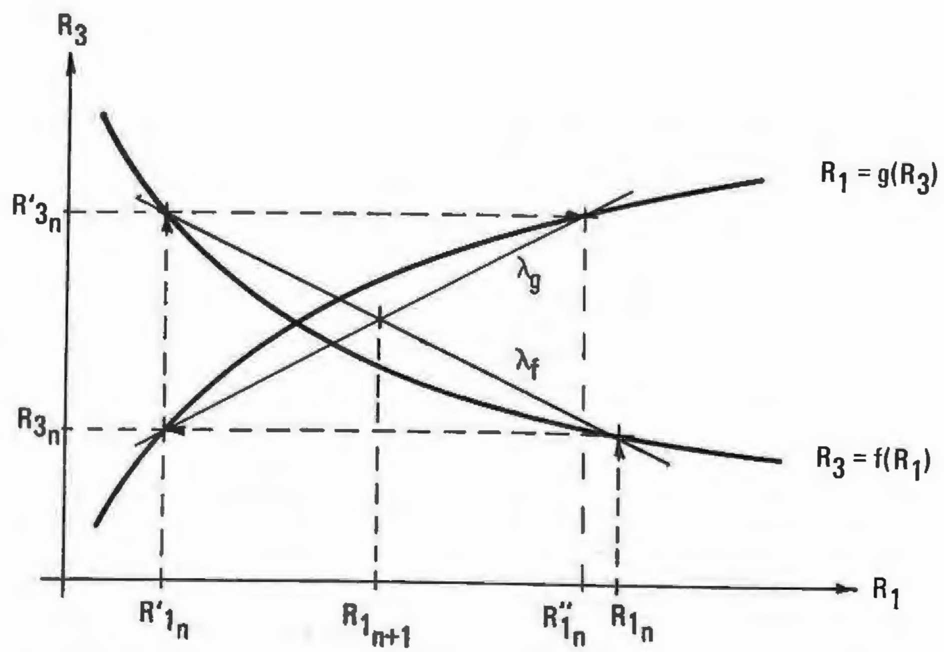


Figure 2-9.4 Esperti's iterative method.

Referring to Fig. 2-9.4, an initial guess for R_1 is made. The corresponding value for R_3 is calculated using $R_3 = f(R_1)$. The corresponding value for R_1 (say R'_1) is calculated using the above value for R_3 in $R'_1 = g(R_3)$. Using $R_3 = f(R'_1)$, a second value of R_3 is calculated; this value of R_3 is designated R'_3. Finally, a second value of R_1 is calculated using $R_1 = g(R'_3)$; this value of R_1 is designated R''_1. Straight lines designated λ_f and λ_g are drawn as shown. The intersection of these two lines defines the next guess for R_1. The iteration is halted when the new and old values for R_1 agree within 10^{-6}%. The convergence of this method is quite fast with four iterations generally providing the above accuracy. Furthermore, the method will converge when direct substitution type iteration proves to be divergent.

The above procedure may be done algebraically to yield a recursion relationship as shown below:

$$R_{1_{n+1}} = R_{1_n} + \frac{g(f(R_{1_n})) - R_{1_n}}{1 - \dfrac{g(f(g(f(R_{1_n})))) - g(f(R_{1_n}))}{g(f(R_{1_n})) - R_{1_n}}} \qquad (2\text{-}9.23)$$

The recursion relationship may be further reduced to an algorithm that can be used to program the HP-97. This algorithm is shown below:

$$R'_{1_n} = g(f(R_{1_n})) \qquad (2\text{-}9.24)$$

$$R''_{1_n} = g(f(R'_{1_n})) \qquad (2\text{-}9.25)$$

$$\delta = R'_{1_n} - R_{1_n} \qquad (2\text{-}9.26)$$

$$\delta' = R''_{1_n} - R'_{1_n} \qquad (2\text{-}9.27)$$

$$\varepsilon = \frac{\delta}{1 - \delta'/\delta} \qquad (2\text{-}9.28)$$

$$R_{1_{n+1}} = R_{1_n} + \varepsilon \qquad (2\text{-}9.29)$$

$$\text{Terminate if } \left|\frac{\varepsilon}{R_{1_{n+1}}}\right| \le 10^{-8} \qquad (2\text{-}9.30)$$

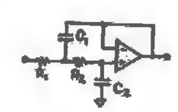
Each time through the $R''_1 = g(f(R_1))$ calculation, the value of R_3 is stored in a scratchpad register. After the iteration loop termination, values for R_1 and R_3 will be at hand. The following formula relates R_2 to these resistors and the other known quantities:

$$R_2 = \frac{C}{C_1 C_2 C_3 R_1 R_3} \qquad (2\text{-}8.31)$$

To simplify the initial guess for R_1 and to keep the range of numbers within bounds, the selected values for the capacitors are normalized to 1 ohm, 1 radian/second values for use by the program. After the corresponding normalized resistors are calculated, the resistance values are de-normalized before output.

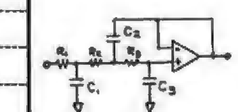
User Instructions

BUTTERWORTH & CHEBYSHEV ACTIVE LP FILTER DESIGN & POLES				
C: n↑ ϵ_{dB}			enter $f_{-\epsilon dB}$ & start	
B: n	B: $-\epsilon_{dB}$	R	enter f_{-3dB} & start	enter $C_1' \uparrow C_2'$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card # 1			
2	If Chebyshev: enter filter order enter passband ripple go to step 4	 n ϵ_{dB}	 [ENT↑] [f] [A]	
3	If Butterworth: enter filter order if bandedge is defined by other than the -3dB point, enter the dB down defining the bandedge	 n $-\epsilon_{dB}$	 [A] [B]	
4	Enter the design resistance level	R, Ω	[C]	
5	If bandedge is -3dB point, enter f-3dB & start	f_{-3dB}, Hz	[D]	if Cheb $f_{-\epsilon dB}$ see below for rest
6	If bandedge is -εdB point, enter $f_{-\epsilon dB}$ & start The data is for the second order filter section, and alternate capacitor values entered in next step are also for the second order section. The third order section (for odd filter order) is output last and is described on the next page.	$f_{-\epsilon dB}$, Hz	[f] [D]  third order section output	if Buttw f_{-3dB} space ω_n Q C_1, F C_2, F stop ω_n Q σ flashing display
7	If alternate capacitor values are desired, enter C_1' } to skip this step, press enter C_2' } "E" without numeric entry After the second resistor value output, the program execution will automatically return to step six until all second order sections have been outputted. If the filter is odd order, the display will flash to indicate the reading of the second card is required.	 C_1', F C_2', F	 [ENT↑] [E]	 R_1, Ω R_2, Ω

User Instructions

BUTTERWORTH & CHEBYSHEV ACTIVE LP FILTERS - ODD ORDER	
THIS SECOND CARD IS USED WITH ODD ORDER BUTTERWORTH AND CHEBYSHEV ACTIVE LP FILTER DESIGN.	enter capacitor changes $C_1' \uparrow C_2' \uparrow C_3'$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
9	Read both sides of second card when display flashes with first program. Program operation will automatically resume after the second side of this card is read.			
10	The three capacitor values for the equal resistor topology will be printed.			C_1, F C_2, F C_3, F
11	If alternate capacitor values are desired, key those values via key "E". If the third order section requirements cannot be met with those resistors, the program execution will halt displaying "ERROR". Press any key to clear the display, and enter another set of capacitors using key "E". By staying close to the capacitor values printed in step 2, the error situation will generally be avoided.	 C_1', F C_2', F C_3', F	 [ENT↑] [ENT↑] [E]	 R_1, Ω R_2, Ω R_3, Ω
12	To run another case, reload both sides of card 1, and return to step 3			

Example 2-9.1

A 1 dB ripple Chebyshev lowpass active filter must pass all frequencies between dc and 1000 Hz within 3 dB, and must reject all frequencies higher than 2000 Hz by more than 60 dB. Program 2-1 may be used to determine the necessary filter order. This program calculates a minimum filter order of 6.19, which is rounded to the next highest integer, 7. A 7th order, 1 dB ripple Chebyshev lowpass filter that is 3 dB down at 1000 Hz, will be 1 dB down at 983.1 Hz and 69.4 dB down at 2000 Hz (λ = 2000/983.1 = 2.035).

This program (Program 2-9) is used to calculate the element values for a 7th order, 1 dB ripple, 1000 Hz -3 dB cutoff frequency Chebyshev filter. A design resistance level of 10000 ohms is chosen which will make the capacitor values around $1/(2\pi fR) = 0.016\ \mu F$.

PROGRAM INPUT

7.	ENT↑	n
1.	GSBa	ϵ_{dB}
10000.	GSBC	design resistance level
1000.	GSBD	-3dB frequency
983.1+00	***	-1dB frequence (output)

PROGRAM OUTPUT

section one			
6.154+03	***	ω_{n_a}	
10.90+00	***	Q_a	
354.2-05	***	C_{1_a}	
745.5-12	***	C_{2_a}	
.47-06	ENT↑	C_{1_a}	alternate values
750.-12	GSBE	C_{2_a}	alternate values
14.83+03	***	R_{1_a}	
5.052+03	***	R_{2_a}	
section two			
4.993+03	***	ω_{n_b}	
3.156+00	***	Q_b	
126.4-09	***	C_{1_b}	
3.173-09	***	C_{2_b}	
.22-06	ENT↑	C_{1_b}	alternate values
3.-09	GSBE	C_{2_b}	alternate values
17.72+03	***	R_{1_b}	
3.429+03	***	R_{2_b}	
section three			
2.965+03	***	ω_n	of second order pair
1.297+00	***	Q	of second order pair
1.269+03	***	σ, real pole location	
85.66-05	***	C_{1_c}	
163.9-05	***	C_{2_c}	
6.385-05	***	C_{3_c}	
.1-06	ENT↑	C_{1_c}	alternate values
.22-06	ENT↑	C_{2_c}	alternate values
6.2-09	GSBE	C_{3_c}	alternate values
8.800+03	***	R_{1_c}	
6.113+03	***	R_{2_c}	
12.22+03	***	R_{3_c}	

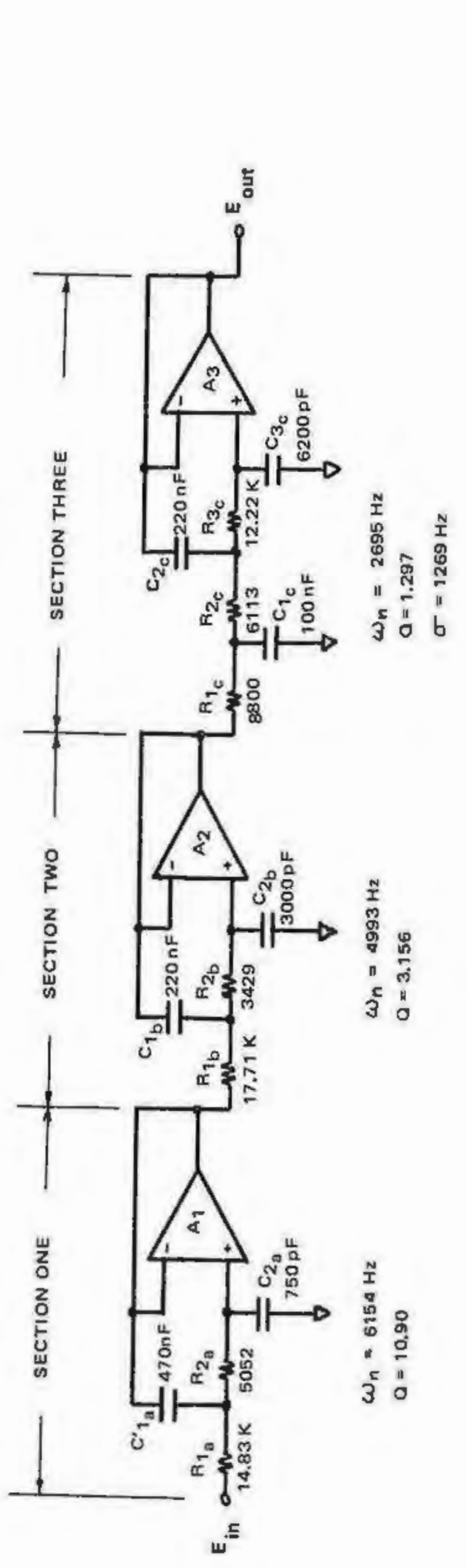


Figure 2-9.5 Overall active filter schematic:

7th order Chebyshev lowpass
1 dB passband ripple
-3 dB at 1000 Hz
-69.4 dB at 2000 Hz

Note: This ordering of the filter sections will result in the lowest output noise assuming the resistance levels in the resonator sections are low enough so the op-amp voltage noise dominates (see Program 1-6). Because the highest Q resonator is first, it will be prone to overload at frequencies near the resonant peak. For filters operating at higher signal levels where self noise is not a concern, the ordering of the sections should be reversed with the lowest Q section placed first.

Example 2-9.2

An active Butterworth lowpass filter must pass all frequencies between dc and 1000 Hz within 1 dB, and must reject all frequencies higher than 3000 Hz by at least 60 dB. Program 2-1 may be used to determine the minimum filter order. This program calculates a minimum filter order of 6.90, which is rounded to 7, the next highest integer. This filter will be 60.9 dB down at 3000 Hz (λ = 3000/1000 = 3).

This program (Program 2-9) is used to find the element values for a 7th order, 1000 Hz -1 dB cutoff, Butterworth lowpass active filter. A design resistance level of 10000 ohms will keep the capacitor values centered around $1/(2\pi fR) = 0.016\ \mu F$.

PROGRAM INPUT

7.	GSBA	n
1.	GSBB	ε_{dB}
10000.	GSBC	R, design resist level
1000.	GSBa	f-εdB
1.101+03	***	f-3dB (output)

PROGRAM OUTPUT

section one

6.920+03	***	ω_n	
2.247+00	***	Q	
64.94-09	***	C_1	
3.216-09	***	C_2	
68.-09	ENT↑	C'_1	alternate values
3000.-12	GSBE	C'_2	alternate values
14.26+03	***	R_1	
7.180+03	***	R_2	

section two

6.920+03	***	ω_n	
801.9-03	***	Q	
23.18-09	***	C_1	
9.010-09	***	C_2	
24.-09	ENT↑	C'_1	alternate values
8200.-12	GSBE	C'_2	alternate values
14.81+03	***	R_1	
7.164+03	***	R_2	

section three

6.920+03	***	ω_n	of second order pair
555.0-03	***	Q	of second order pair
6.920+03	***	σ	
19.32-09	***	C_1	
22.14-09	***	C_2	
7.058-09	***	C_3	
22.-09	ENT↑	C'_1	alternate values
22.-09	ENT↑	C'_2	alternate values
6800.-12	GSBE	C'_3	alternate values
9.172+03	***	R_1	
7.675+03	***	R_2	
13.63+03	***	R_3	

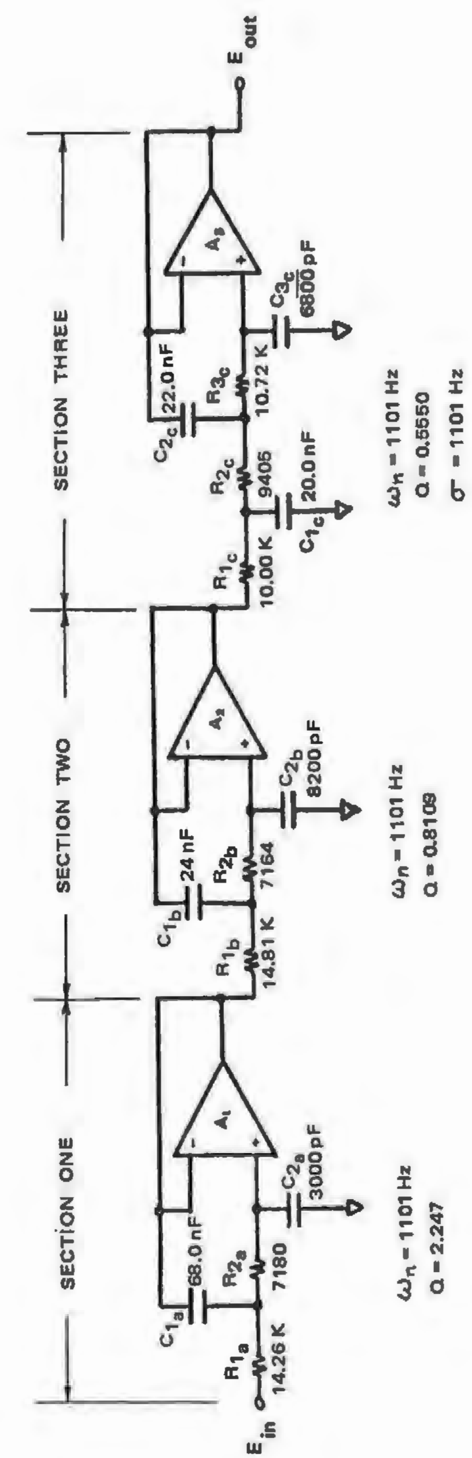


Figure 2-8.6 Overall active filter schematic:
7th order Butterworth lowpass filter
-1 dB @ 1000 Hz
-3 dB @ 1101 Hz
-60.9 dB @ 2000 Hz

Program Listing I

Key	Comment
*LBLA	BUTTERWORTH; LOAD n
SF1	indicate Butterworth
EEX	setup registers:
STOB	$f_{-3dB}/f_{-\epsilon dB} = 1$
STOD	cosh a = 1
STOE	sinh a = 1
R↓	recover n
GTO8	goto filter order entry subr.
*LBLa	CHEBYSHEV: LOAD n↑εdB
STOB	store εdB
R↓	recover filter order, n
CF1	indicate Chebyshev
GSB8	gosub filter order entry subr
RCLB	
EEX	
1	calculate:
÷	
10ˣ	$\epsilon = \sqrt{10^{0.1\epsilon_{dB}} - 1}$
EEX	
-	
√X	
1/X	$1/\epsilon \rightarrow R5$
STO5	
ENT↑	
X²	
EEX	calculate and store:
+	
√X	$a = \frac{1}{n} \sinh^{-1}(\frac{1}{\epsilon}) \rightarrow R2$
+	
RCLA	
1/X	
Yˣ	
STO2	
ENT↑	
1/X	
-	calculate and store:
2	
÷	sinh a $\rightarrow R_E$
STOE	
RCL2	
ENT↑	
1/X	calculate and store:
+	
2	cosh a $\rightarrow R_D$
÷	
STOD	
GTO5	go to data entry flag clear
*LBLB	LOAD $-\epsilon_{dB}$ FOR BUTTERWORTH
EEX	if bandedge is not defined
1	by -3dB point
÷	calculate and store:
10ˣ	
EEX	$\frac{f_{-3dB}}{f_{-\epsilon dB}} = \left[10^{0.1\epsilon_{dB}} - 1\right]^{\frac{1}{2n}} \rightarrow R_B$
-	
RCLA	
ENT↑	
+	
1/X	
CHS	
Yˣ	
STOB	
GTO5	goto data entry flag clear
*LBLC	LOAD OPERATING RESISTANCE
STO6	LEVEL
GTO5	goto data entry flag clear
*LBLD	LOAD f_{-3dB} & START
F1?	
GTO7	jump, if Butterworth
STO9	
RCL5	
ENT↑	
X²	
EEX	
-	
√X	
+	for Chebyshev, calculate
RCLA	
1/X	$f_{-\epsilon dB} = \frac{f_{-3\ dB}}{\cosh\left[\frac{1}{n}\cosh^{-1}(\frac{1}{\epsilon})\right]}$
Yˣ	
ENT↑	
1/X	
+	
÷	
ENT↑	
+	
SF3	set data entry flag
PRTX	print $f_{-\epsilon dB}$
*LBLd	LOAD $f_{-\epsilon}$ dB AND START
F1?	
RCLB	if Butterworth, calculate
F1?	and print f_{-3dB}
x	
F1?	
PRTX	
*LBL7	
Pi	
ENT↑	
+	
x	if flag 3, $2\pi f \rightarrow R3$
F3?	
STO3	
EEX	
STO0	setup for next loop
SPC	
*LBL1	second order filter loop
SPC	
RCL0	
RCLI	
x	
EEX	

REGISTERS

0	1	2	3	4	5	6	7	8	9
2k - 1	2Q	a or ω_n	ω_{-3dB}	C_1	$1/\epsilon$	R	ω_k	σ_k	scratch
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
filter order, n	ϵ_{dB}, 1, or $\frac{f_{-3dB}}{f_{-\epsilon dB}}$	C_2	cosh a or 1	sinh a or 1	$\pi/(2n)$

Program Listing II

Key	Comment
→R	
RCLD	calculate pole positions:
x	
STO7	$\sigma_k = (\sinh a)(\sin \frac{2k-1}{2n}\pi)$
X⇄Y	
RCLE	$\omega_k = (\cosh a)(\cos \frac{2k-1}{2n}\pi)$
x	
STO8	
→P	
RCL3	
x	calculate ω_{n_k} and Q_k
PRTX	
STO2	$\omega_{n_k}^2 = \sigma_k^2 + \omega_k^2$
X⇄Y	
COS	$Q_k = 1/(2 \cos(\tan^{-1} \frac{\omega_k}{\sigma_k}))$
1/X	
STO1	
2	
÷	
PRTX	
LSTX	increment k by 2
ST+0	
F0?	jump if n is even
GTO3	
RCL0	
EEX	odd order filter:
+	test for last section
RCLA	(3rd order section)
X≤Y?	
GTO2	
*LBL3	calculate C_1 for 2nd order
RCL1	
RCL2	
÷	$C_1 = \frac{2Q}{\omega_n \cdot R}$
RCL6	
÷	
PRTX	
ENT↑	save C_1 in stack
ENT↑	
RCL1	calculate C_2 for 2nd order
X²	
÷	$C_2 = \frac{C_1}{4 \cdot Q^2}$
PRTX	
RTN	
*LBLE	LOAD ALT CAPACITOR VALUES
F3?	
F3?	if numeric entry, store,
GTO4	otherwise jump
STO9	
X⇄Y	calculate R_1:
STO4	
÷	
RCL1	$R_1 = \frac{1 + \sqrt{1 - 4Q^2 \cdot C_2'/C_1'}}{2 \cdot Q \cdot \omega_n \cdot C_2'}$
X²	
x	
CHS	
EEX	R_1 calculation (continued)
+	
√X	
EEX	
+	
RCL1	
÷	
RCL2	
÷	
RCL9	
÷	
PRTX	
RCL4	R_2 calculation:
x	
RCL9	
x	
RCL2	$R_2 = 1/(\omega_n^2 \cdot C_1' \cdot C_2' \cdot R_1)$
X²	
x	
1/X	
PRTX	
*LBL4	
RCL0	test for loop exit
RCLA	
X>Y?	
GTO1	
SPC	space paper upon loop exit
SPC	
RTN	
*LBL2	calculate real 3rd order pole
RCLE	
RCL3	
x	
PRTX	
*LBL0	wait loop for 2nd card read
PSE	
GTO0	
*LBL8	filter order entry subr
STOA	
2	
÷	
ENT↑	set flag 0 if n is even
INT	
CF0	
X=Y?	
SF0	
Pi	
RCLA	calculate and store:
ENT↑	
+	$\pi/(2n) \rightarrow R_I$
÷	
STOI	
*LBL5	data entry flag clear subr
CF3	
RTN	

LABELS

A	B	C	D	E
load n ↑ dB	load R	f_{-3dB} & go	$f_{-\epsilon dB}$ & go	load C_1↑C_2
a capacitor entry tog	b	c	d	e
0 2nd card read loop	1 n & Q calculation	2 odd order output	3 even ord output	4 loop exit test
5 clr data entry flag	6	7	8	9

FLAGS

Flag	Use
0	n even
1	set for Buttr
2	
3	data entry

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0		■	DEG		FIX	
1	■		GRAD		SCI	
2		■	RAD	■	ENG	■
3		■			n	3

Program Listing I

Step	Key	Comments
001	RCL7	ω_k for 3rd order section
002	RCL8	σ_k
003	RCL6	R
004	P⇄S	
005	SF2	
006	STO6	store denormalization R
007	R↓	
008	STO0	calculate and store:
009	→P	$\omega_{n_k}{}^2 = \omega_k{}^2 + \sigma_k{}^2$
010	X²	
011	STO1	
012	2	$2\sigma_k = \frac{\omega_n}{Q}$
013	ST×0	
014	RCLE	calculate and store:
015	RCL1	
016	×	$C = \frac{\tau}{\omega_n^2}$
017	1/X	
018	STOC	
019	STO5	
020	RCLE	calculate and store:
021	RCL0	
022	+	$B = \frac{\tau}{\omega_n \cdot Q} + \frac{1}{\omega_n^2}$ as
023	×	
024	STOB	$(\frac{\tau}{\omega_n^2})(\frac{1}{\tau} + \frac{\omega_n}{Q})$
025	STO4	
026	RCLE	calculate and store:
027	RCL0	
028	×	$A = \tau + \frac{1}{\omega_n Q}$ as
029	RCL1	
030	+	
031	RCLC	$(\frac{\tau}{\omega_n^2})(\omega_n^2 + \frac{\omega_n}{Q})$
032	×	
033	STOA	
034	3	use register arithmetic
035	ST×4	to form and store:
036	ST×5	3B/2, and 3C
037	2	
038	ST÷4	
039	STO3	initialize registers for
040	EEX	Newton-Raphson iteration
041	CHS	
042	8	
043	STO9	
044	*LBL0	Newton-Raphson routine to
045	RCL3	find the real 3rd order
046	RCL3	root of $f(C_1) = 0$
047	RCL3	
048	RCLA	
049	-	calculate $f(C_1)$ as:
050	×	
051	RCL4	$f(C_1) = C_1^3 - AC_1^2 + \frac{3B}{2}C_1 - 3C$
052	+	
053	×	
054	RCL5	
055	-	
056	STO0	
057	CLX	calculate $f'(C_1)$ as
058	3	
059	×	$f'(C_1) = 3C_1^2 - 2AC_1 + \frac{3B}{2}$
060	RCLA	
061	2	
062	×	
063	-	
064	×	
065	RCL4	
066	+	
067	ST÷0	form and store:
068	RCL0	$C_{1_{n+1}} = C_{1_n} - \frac{f(C_{1n})}{f'(C_{1n})}$
069	ST-3	
070	RCL3	iterate again if
071	÷	
072	ABS	$\left\| \frac{\frac{f(C_{1n})}{f'(C_{1n})}}{C_{1n}} \right\| \geqslant 10^{-8}$
073	RCL9	
074	X≤Y?	
075	GTO0	
076	RCLA	calculate and store:
077	RCL3	
078	-	
079	3	$C_3 = \frac{A - C_1}{3}$
080	÷	
081	STO4	
082	RCL3	calculate:
083	×	
084	RCLC	$C_2 = \frac{C}{C_1 \cdot C_3}$
085	X⇄Y	
086	÷	
087	RCL4	order C_1, C_2, and C_3 in
088	X⇄Y	the stack
089	RCL3	
090	GSB9	restore P S order
091	GSB1	denormalize and print
092	GSB1	capacitors
093	*LBL1	
094	RCL3	capacitor denormalization
095	÷	
096	RCL6	$C_{den} = C_{nor}/(\omega_{\epsilon dB} \cdot R)$
097	÷	
098	PRTX	
099	R↓	
100	RTN	
101	*LBLE	LOAD ALTERNATE CAPACITOR
102	GSB9	VALUES FOR C_1, C_2, & C_3
103	RCL3	
104	P⇄S	
105	SF2	initialize registers
106	R↓	
107	STO2	store C_1, C_2, & C_3
108	R↓	
109	STO1	
110	R↓	

REGISTERS

0 2k-1	1	2	3 ω_{-3dB}	4	5 $1/\epsilon$	6 R	7 ω_k	8 σ_k	9
S0 C_1', $\frac{\omega_n}{Q}$	S1 C_2', $\omega_n{}^2$	S2 C_3'	S3 R_1	S4 R_1', $\frac{3B}{2}$	S5 R_1'', 3C	S6 R	S7 scratch	S8 scratch	S9 10^{-8}

A $A = \tau + \frac{1}{\omega_n Q}$	B $B = \frac{\tau}{\omega_n Q} + \frac{1}{\omega_n^2}$	C $C = \frac{\tau}{\omega_n^2}$	D cosh a	E sinh a	I $C/(C_1 \cdot C_2)$

Program Listing II

Step	Key	Comments
111	STO0	
112	R↓	
113	RCL6	form $\omega_{\epsilon dB}$ * R
114	×	
115	ST×0	
116	ST×1	normalize C_1', C_2', C_3'
117	ST×2	
118	EEX	store initial guess for R_1
119	STO3	
120	RCLC	
121	RCL0	form and store: $\frac{C}{C_1' \cdot C_2'}$
122	÷	
123	RCL1	
124	÷	
125	STOI	
126	*LBL7	Esperti iteration loop start
127	RCL3	calculate and store:
128	GSB8	$R_1' = f(g(R_1))$
129	STO4	
130	GSB8	calculate and store
131	STO5	$R_1'' = f(g(R_1'))$
132	RCL4	calculate: $\delta = R_1' - R_1$
133	ST-5	
134	RCL3	calculate: $\delta' = R_1'' - R_1'$
135	ST-4	
136	RCL4	
137	X=0?	$\delta = 0$ escape
138	GTO6	
139	EEX	
140	RCL5	
141	RCL4	form: $\Delta R_1 = \frac{\delta}{1 - \delta'/\delta}$
142	÷	
143	-	
144	÷	
145	ST+3	form $R1_{n+1} = R1_n + \Delta R1$
146	RCL3	
147	÷	iterate again if
148	ABS	
149	RCL9	$\left\| \frac{\Delta R_1}{R_1} \right\| \geqslant 10^{-8}$
150	X≤Y?	
151	GTO7	
152	*LBL6	
153	RCLI	
154	RCL2	
155	÷	calculate $R_2 = \frac{C}{C_1 C_2 C_3 R_1 R_3}$
156	RCL3	
157	÷	
158	RCL8	
159	÷	
160	RCL6	
161	ST×3	denormalize resistors
162	ST×8	
163	×	
164	RCL3	print R_1, R_2, and R_3
165	PRTX	
166	X⇄Y	
167	PRTX	
168	RCL8	
169	PRTX	
170	*LBL9	
171	F2?	if flag 2, restore
172	P⇄S	P⇄S register order
173	RTN	
174	*LBL8	subroutine for $R1 = f(g(R_1))$
175	RCL0	
176	RCL0 C_1	$R_3 = g(R_1)$
177	RCL2 C_3	as defined by:
178	+	
179	STO7	$R_3 = \frac{R_1^2C_1(C_1+C_3) - R_1AC_1 + R_1B - \frac{C}{C_1}}{R_1^2C_2C_3}$
180	×	
181	×	
182	RCLA A	
183	RCL0 C_1	
184	×	
185	-	
186	×	
187	RCLB B	
188	+	
189	RCLC C	
190	RCL0 C_1	
191	÷	
192	-	
193	RCL1 C_2	
194	RCL2 C_3	
195	×	
196	÷	
197	X⇄Y	
198	X²	
199	÷	
200	STO8	store R_3
201	RCL2 C_3;	$R_1 = f(R_3)$ as defined by:
202	×	
203	RCLA A	$R_1 = -\frac{b}{2a} + \sqrt{\left[\frac{b}{2a}\right]^2 - \frac{c}{a}}$
204	-	
205	CHS	
206	RCL7	
207	ENT↑	$a = C_1 + C_3$
208	+	
209	÷ $-\frac{b}{2a}$	$-b = A - C_3R_3$
210	ENT↑	
211	X²	$c = C/(C_1C_2R_3)$
212	RCLI	
213	RCL8	
214	÷	
215	RCL7	
216	÷ $\frac{c}{a}$	
217	-	
218	√X	
219	+ R_1	
220	RTN	

LABELS

A	B	C	D	E load C_1, C_2, C_3
a	b	c	d	e
0 Newton-Raphson	1 denormalize routine	2	3	4
5	6 escape	7 Esperti routine go	8 $R_1 = f(g(R_1))$	9 P⇄S routine

FLAGS

0	
1	
2	P⇄S used odd # time
3	

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0			DEG		FIX	
1			GRAD		SCI	
2		■	RAD	■	ENG	■
3					n 3	

HP-67 suggested program changes. Program space does not allow the addition of a print, R/S toggle and associated output routine. If the HP-67 user would like the program to stop instead of halting for 5 seconds (print command) change the "print" statements to "R/S" at the following line numbers: (program 1); 122, 130, 147, 153, 178, 187, and 200; (program 2); 098, 165, 167, and 169. To resume program execution with the above changes, execute a "R/S" command from the keyboard after each data output point.

PROGRAM 2-10 BUTTERWORTH AND CHEBYSHEV ACTIVE HIGHPASS FILTER DESIGN AND POLE LOCATIONS.

Program Description and Equations Used

This program calculates the normalized pole locations and provides element values for the un-normalized, unity gain Sallen and Key type second and third order highpass active resonator circuit. Higher order filters are formed by cascading second order sections, and one third order section if the filter order is odd. The program uses either the Butterworth (maximally flat) or Chebyshev (equiripple passband) all pole filter descriptions.

The program is designed to allow the use of specified capacitor values such as would result from the actual measurement of a standard value capacitor. The corresponding resistor values are calculated for each section. The nearest 1% standard value precision resistor will generally suffice for the calculated value.

The design process starts by finding the normalized lowpass pole locations for the desired filter type. If the passband cutoff frequency is different from the conventional definition of the bandedge, a scaling of the normalized cutoff frequency is done. The Butterworth amplitude response is 3 dB down at the passband edge, while the Chebyshev amplitude response is ε dB down at the passband edge, where ε dB is the passband ripple in dB. The scaling factor is K, and the normalized filter cutoff frequency is denoted by ω_n.

The normalized and scaled lowpass pole locations are sequentially found as complex conjugate pairs, and, if the filter order is odd, the real pole location. The lowpass, unity-gain, Sallen and Key, normalized active filter circuit element values may be found in terms of these pole locations. The element values of the highpass normalized active resonator may be found from the normalized lowpass structure. The normalized lowpass structure is transformed to the normalized highpass structure by replacing each lowpass resistor with a capacitor and vice versa.

The normalized highpass element values are the reciprocals of the corresponding converted lowpass element, i.e., a 2 farad capacitor becomes a ½ ohm resistor. This conversion is equivalent to replacing s by 1/s in the lowpass transfer function equation. The un-normalized highpass equation is found by replacing s by ω_c/s, where $\omega_c = 2\pi f_c$, and f_c is the highpass cutoff frequency in hertz.

Each complex conjugate pole pair can be expressed in either the cartesian (real and imaginary parts) or the polar (magnitude and angle) co-ordinate system. A variation on the polar system allows the pole pair to be defined in terms of the natural frequency, ω_n, and "Q" or quality factor. The relationships between these co-ordinate systems is shown in Fig. 2-9.1 The Butterworth and Chebyshev pole locations are given in Program 2-2. By putting all the foregoing concepts together, the denormalized highpass element values can be expressed in terms of ω_n and Q with the second order circuit topology as shown in Fig. 2-10.1.

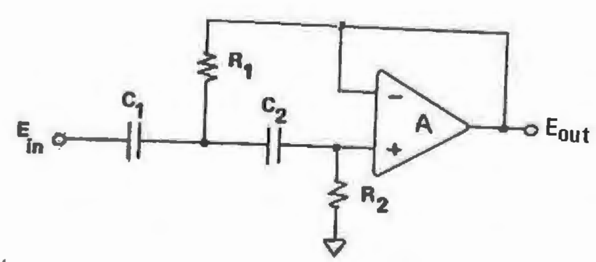


Figure 2-10.1 Highpass Sallen and Key circuit.

$$R_1 = \frac{\omega_n/\omega_c}{Q(C_1 + C_2)} \qquad (2\text{-}10.1)$$

$$R_2 = \frac{(\omega_n/\omega_c)^2}{R_1 \cdot C_1 \cdot C_2} \qquad (2\text{-}10.2)$$

The Sallen and Key unity-gain op-amp resonator is chosen over other types because of its low component count and low parameter sensitivities to element value changes (see [19]). High Q realizations are difficult with this resonator type since the resistor value spread is $4Q^2$ when the capacitor values are equal, however, this constraint is not a problem here since the pole Q's are rarely greater than 10.

High pole Q's occur with higher order filters (n greater than 9 or so). In these cases, the Szentirmai leapfrog topology [48], should be given consideration, or else an elliptic response lower order filter might meet the amplitude response requirements (the phase response will be less linear however).

All operational amplifiers have bandwidth limitations, i.e., the µA-741 has unity open loop gain at 500 kHz typically. When the operating frequency range of the active filter contains frequencies that approach 1% of the op-amp unity gain crossover frequency (500 kHz for the µA-741), then the contribution of the operational amplifier compensation pole and lower open loop gain must be considered. Program 1-3 can be used to calculate the pole location shifts. Positive and negative feedback resonators of the Deliyannis type can accommodate the op-amp compensation pole and open loop gain characteristic (see [19]).

If the filter order is odd, then a real pole exists. A third order op-amp active resonator circuit may be used to produce both the real pole and a complex conjugate pair. The lowest Q pole pair is chosen for realization by this circuit to keep the element value spread within bounds, and also to minimize sensitivities. The third order active highpass topology is shown in Fig. 2-10.2.

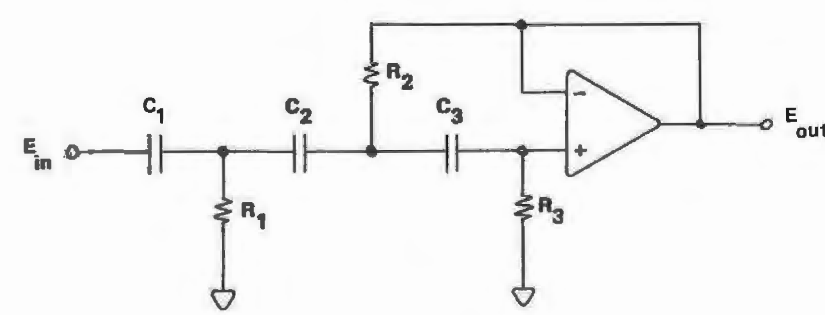


Figure 2-10.2 Third order highpass active filter section.

The transfer function in terms of the R's and C's assuming an ideal operational amplifier is:

$$\frac{E_{out}}{E_{in}} = \frac{s^3 R_1 R_2 R_3 C_1 C_2 C_3}{D(s)} \qquad (2\text{-}10.3)$$

where

$$D(s) = s^3 R_1 R_2 R_3 C_1 C_2 C_3 + s^2 R_2 \{R_3 C_2 C_3 + R_1\ \Sigma CC\}$$

$$+ s\{R_1 (C_1 + C_2) + R_2 (C_2 + C_3)\} + 1$$

and

$$\Sigma CC = C_1 C_2 + C_2 C_3 + C_1 C_3 \qquad (2\text{-}10.4)$$

The resistor values may be obtained from the capacitor values and the pole locations by the simultaneous solution of three equations in three unknowns. These three equations are generated by equating like powers of s between the desired transfer function as expressed with the pole locations and the above transfer function. The desired transfer function in terms of the complex conjugate pole pair as expressed through ω_n and Q, and the real pole location, $1/\tau$, is:

$$\frac{E_{out}}{E_{in}} = \frac{s^3 \left(\frac{1}{\omega_c \tau}\right) \cdot \left(\frac{\omega_n}{\omega_c}\right)^2}{\left\{\frac{s}{\omega_c \tau} + 1\right\}\left\{s^2 \left(\frac{\omega_n}{\omega_c}\right)^2 + s\left(\frac{1}{Q}\right)\left(\frac{\omega_n}{\omega_c}\right) + 1\right\}} \qquad (2\text{-}10.5)$$

or, in descending powers of s:

$$\frac{E_{out}}{E_{in}} = \frac{s^3 \left(\frac{\omega_n}{\omega_c}\right)^2 \cdot \left(\frac{1}{\omega_c \tau}\right)}{s^3 \left(\frac{\omega_n}{\omega_c}\right)^2 \left(\frac{1}{\omega_c \tau}\right) + s^2 \left(\frac{\omega_n}{\omega_c}\right)^2 \left(1 + \frac{1}{\omega_n Q \tau}\right) + s\left(\frac{\omega_n}{\omega_c}\right)\left(\frac{1}{Q} + \frac{1}{\omega_n \tau}\right) + 1} \qquad (2\text{-}10.6)$$

The resulting three equations in three unknowns are:

$$R_1 R_2 R_3 C_1 C_2 C_3 = \left(\frac{\omega_n}{\omega_c}\right)^2 \left(\frac{1}{\omega_c \tau}\right) \qquad (2\text{-}10.7)$$

$$R_2 \ (R_3 C_2 C_3 + R_1 \Sigma CC) = \left(\frac{\omega_n}{\omega_c}\right)^2 \left(1 + \frac{1}{\omega_n Q \tau}\right) \qquad (2\text{-}10.8)$$

$$R_1 \ (C_1 + C_2) + R_2 (C_2 + C_3) = \left(\frac{\omega_n}{\omega_c}\right)\left(\frac{1}{Q} + \frac{1}{\omega_n \tau}\right) \qquad (2\text{-}10.9)$$

After algebraic manipulation, a cubic equation in R_1 alone is obtained:

$$R_1^3 K_3 + R_1^2 K_2 - R_1 K_1 + K_0 = 0 \qquad (2\text{-}10.10)$$

where the constants K_3, K_2, K_1, and K_0 are defined by:

$$K_3 = -\ (C_1 + C_2)(C_1\ \Sigma CC) \qquad (2\text{-}10.11)$$

$$K_2 = \left(\frac{1}{Q} + \frac{1}{\omega_n \tau}\right)\left(C_1 \Sigma CC\right)\left(\frac{\omega_n}{\omega_c}\right) \qquad (2\text{-}10.12)$$

$$K_1 = \left(1 + \frac{1}{\omega_n Q \tau}\right)\left(C_1 (C_2 + C_3)\right)\left(\frac{\omega_n}{\omega_c}\right)^2 \qquad (2\text{-}10.13)$$

$$K_0 = (C_2 + C_3)\left(\frac{1}{\omega_c \tau}\right)\left(\frac{\omega_n}{\omega_c}\right)^2 \qquad (2\text{-}10.14)$$

The program uses a Newton-Raphson iterative solution to find the real root of Eq. (2-10.10) for R_1 (there will be at least one real root). The details of the Newton-Raphson technique are shown in Program 1-5.

Once R_1 has been obtained, the values for R_2 and R_3 are obtained using the following equations:

$$R_2 = \left(\frac{1}{C_2 + C_3}\right)\left\{\frac{\omega_n}{\omega_c}\left(\frac{1}{Q} + \frac{1}{\omega_n \tau}\right) - R_1 (C_1 + C_2)\right\} \qquad (2\text{-}10.15)$$

$$R_3 = \left(\frac{\omega_n}{\omega_c}\right)^2 \left(\frac{1}{\omega_c \tau}\right)\left(\frac{1}{R_1 R_2 C_1 C_2 C_3}\right) \qquad (2\text{-}10.16)$$

BUTTERWORTH AND CHEBYSHEV ACTIVE HIGHPASS FILTERS				
C: n↑ε dB			load f-ε dB & start	
B: n	B: ε dB	R	load f-3dB & start	load C_1↑C_2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Read both sides of program card one			
2	If Chebyshev response is desired:			
	a) Load filter order	n	ENT↑	
	b) Load passband ripple in dB	ε dB	f A	
	c) go to step 4			
3	If Butterworth response is desired: Load filter order	n	A	
	If the passband edge is defined at other than the -3dB point, enter the bandedge attenuation in dB (attenuation is expressed as a positive number)	ε dB	B	
4	Load operating resistance level *** The calculated resistor values will usually be within a decade of this value.	R	C	
5	If the passband edge is defined by the -3dB amplitude response point, enter f-3dB ***	f-3dB	D	f-εdB*
	*The Chebyshev bandedge is usually defined by the -εdB point since the passband response oscillates within a band εdB wide. If a Chebyshev response has been selected, the frequency where the amplitude response exits the ε dB ripple band will be printed.			See step6 continuation on next page for rest of output.
	go to step 7 (read step 6 commentary)			
6	If the passband edge is defined by the -εdB point, enter f-εdB ***	f-εdB	f D	f-3dB**
	**If Butterworth response has been selected, the frequency where the response is 3dB down will be printed.			

2-10 **User Instructions**

BUTTERWORTH AND CHEBYSHEV ACTIVE HIGHPASS FILTERS
CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
6				C_{design}
	The design capacitance value is outputted.*** If this value is unacceptable from a circuit or practicality point of view, alter the design resistance level accordingly using key "C", then recalculate the design capacitance level using key "D". The design cutoff frequency need not be re-entered even though the original frequency entry was via keys "f", "D".	new R	C	
			D	new C_{des}
	When an acceptable design capacitance level has been found, continue program output by using "R/S".		R/S	ω_{n1}
				Q_1
				stop
7	Enter capacitor values to be used in this second order filter section***	C_1	ENT↑	
		C_2	E	R_1
				R_2
				space
				n2
				Q_2
				stop
	Keep entering capacitor values for succeeding sections until all second order sections have been defined.	$C1_2$		$R1_2$
		$C2_2$		⋮
	If an odd order filter is being designed, the last printout will be a set of three numbers, and the display will flash to indicate that the loading of the second card is required. It is not necessary to stop the program, just insert the second card into the card reader and read both sides.	⋮		⋮
		$C1_n$		
		$C2_n$		$R2_n$
				odd order filter: last sect
	After the second card reading is complete, load the three capacitor values to be used with this third order filter section using key "E".			ω_n
				Q
				1/τ
				flashing display
		C_1	ENT↑	
	*** The unit of resistance is ohms,	C_2	ENT↑	
	capacitance is farads, and frequency	C_3	E	R_1
	is hertz.			R_2
				R_3

Example 2-10.1

A fifth order, ½ dB passband ripple Chebyshev active highpass filter is to have 3 dB or less attenuation at 10 Hz. A National Semiconductor type LF-156 bi-fet operational amplifier is chosen as the active element in the filter.

Design an active filter to meet these specifications and choose the operating resistance level to achieve the lowest capacitance values in the filter without affecting the dc drift characteristics of the operational amplifier by more than 10%. The operating temperature range is -25°C to +85°C.

From the LF-156 data sheet, the maximum input bias current occurs at the highest operating temperature, +85°C, and is approximately 1 nA. The typical input offset voltage is 3 millivolts. The resistance level that will generate 0.3 millivolts with 1 nA flowing is:

$$R = (3 \times 10^{-4}\ \text{V})/(1 \times 10^{-9}\ \text{A}) = 300\ \text{k}\Omega$$

The filter is then designed with this value in mind as the largest resistance value which has an effect on the dc output of the last filter stage. Being a highpass filter, each stage of the filter blocks the dc voltage present from the preceding stage.

The filter design will be done twice, once with 300 kΩ as the design resistance level to determine the value of R_2 in the last (third order) section. The operating resistance level is then scaled to cause the highest resistance value (R_2) to be 300 k Ω. The HP-97 printout for these operations is shown on the next page.

In the second run of the program, the design capacitance level is 0.1749 μF. The nearest larger standard capacitor value is 0.22 μF. The filter will require five capacitors, therefore, five 0.22 μF mylar capacitors were drawn from stock, and their capacities measured. The measured values were: .2236 μF, .2014 μF, .1965 μF, .2173 μF, and 0.2542 μF. The filter resistances are designed around these capacitor values.

Example 2-10.1 printout

FIRST PROGRAM RUN

LOAD FIRST PROGRAM CARD

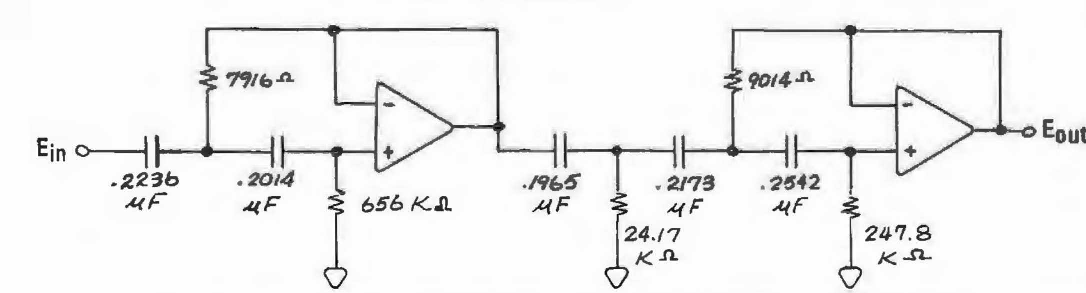
```
      5. ENT↑   load filter order
      .5 GSBa   load passband ripple

 300000. GSBC   load design resist

     10. GSBD   load -3dB frequency
10.59+00  ***   -½dB frequency (o/p)

53.05-09  ***   design capacitance
                level (output)
          R/S   continue execution

960.8-03  ***   ωn  first section
4.545+00  ***   Q
53.05-09 ENT↑   enter first section
         GSBE   design capacitance
31.71+03  ***   R1  first section
2.620+06  ***   R2  resistor values

651.9-03  ***   ωn
1.178+00  ***   Q   second section
342.1-03  ***   1/τ
                LOAD SECOND CARD
53.05-09 ENT↑
         ENT↑     enter design cap
         GSBE

90.47+03  ***   R1 } second section
43.86+03  ***   R2 } resistor values
989.1+03  ***   R3 }

989.1+03 ENT↑
 300.+03   ÷      scale design
          1/x     resistance level
303.3-03  ***     to make R3 become
 300000.   ×      300 kΩ
90.99+03  ***
```

SECOND PROGRAM RUN

LOAD FIRST PROGRAM CARD

```
       5. ENT↑
       .5 GSBa
90.99+03 GSBC   load new design
                resistance level
      10. GSBD
10.59+00  ***

174.9-03  ***   new design capacitor
                value (output)
          R/S

960.8-03  ***   ωn
4.545+00  ***   Q
.2236-06 ENT↑   C1  first section
.2014-06 GSBE   C2  selected caps
7.916+03  ***   R1  first section
655.9+03  ***   R2  resistor values

651.9-03  ***   ωn
1.178+00  ***   Q   second section
342.1-03  ***   1/τ
                LOAD SECOND CARD

.1965-06 ENT↑   C1 } second section
.2173-06 ENT↑   C2 } input (third
.2542-06 GSBE   C3 } order filter)

24.17+03  ***   R1 } second section
9.014+03  ***   R2 } resistor values
247.8+03  ***   R3 }
```

FINAL SCHEMATIC



Program Listing I

Step	Key	Comment
001	*LBLA	BUTTERWORTH: LOAD n
002	SF1	indicate Butterworth
003	EEX	setup registers:
004	STOB	f-3dB/f-εdB = 1
005	STOD	cosh a = 1
006	STOE	sinh a = 1
007	GSB5	
008	GTO6	
009	*LBLa	CHEBYSHEV: LOAD n ↑ εdB
010	CF1	indicate Chebyshev
011	STOB	store εdB
012	GSB5	gosub input routine
013	RCLB	calculate:
014	EEX	
015	1	$\epsilon = (10^{0.1\epsilon dB} - 1)^{\frac{1}{2}}$
016	÷	
017	10ˣ	
018	EEX	
019	-	
020	√X	
021	1/X	store $1/\epsilon \rightarrow R5$
022	STO5	
023	ENT↑	calculate and store:
024	X²	
025	EEX	
026	+	
027	√X	$a = \frac{1}{n}\sinh^{-1}(\frac{1}{\epsilon}) \rightarrow R2$
028	+	
029	RCLA	
030	1/X	
031	Yˣ	
032	STO2	
033	ENT↑	calculate and store:
034	1/X	
035	-	$\sinh a \rightarrow RE$
036	2	
037	÷	
038	STOE	
039	RCL2	calculate and store:
040	ENT↑	
041	1/X	
042	+	$\cosh a \rightarrow RD$
043	2	
044	÷	
045	STOD	
046	LSTX	calculate and store:
047	RCL5	
048	ENT↑	
049	X²	
050	EEX	$\frac{f\text{-}\epsilon dB}{f\text{-}3dB} = \cosh(\frac{1}{n}\cosh^{-1}(\frac{1}{\epsilon}))$
051	-	
052	√X	
053	+	
054	RCLA	
055	1/X	
056	Yˣ	
057	ENT↑	
058	1/X	
059	+	
060	÷	
061	STOB	
062	GTO6	
063	*LBLB	LOAD εdB for Butterworth
064	EEX	calculate and store:
065	1	
066	÷	
067	10ˣ	
068	EEX	
069	-	$\frac{f\text{-}3dB}{f\text{-}\epsilon dB} = \left[10^{0.1\epsilon dB} - 1\right]^{\frac{1}{2n}}$
070	RCLA	
071	1/X	
072	Yˣ	
073	√X	
074	1/X	
075	STOB	
076	GTO6	
077	*LBLC	LOAD OPERATING RESISTANCE LEVEL
078	STO6	
079	GTO6	
080	*LBLD	LOAD f-3dB and START
081	STOC	temporarily store f-3dB
082	F1?	jump if Butterworth
083	GTO0	
084	RCLB	recall Cheb denorm ratio
085	STO3	
086	÷	form -εdB frequency
087	F3?	print f-εdB if data entered
088	GSB4	
089	RCLC	recall f-3dB
090	*LBLd	LOAD f-εdB and START
091	STOC	temporarily store frequency
092	RCLB	recall Buttr denorm ratio
093	F1?	if Buttr, store ratio
094	STO3	
095	÷	
096	F1?	if Butterworth, calculate and print f-3dB
097	PRTX	
098	*LBL0	
099	SPC	
100	CF2	
101	RCLC	
102	ENT↑	if flag 3, $2\pi f_c \rightarrow R5$
103	+	
104	Pi	
105	x	
106	F3?	
107	STO5	
108	RCL5	
109	RCL6	calculate and print nominal capacitor value
110	x	

REGISTERS

0	1	2	3	4	5	6	7	8	9
2k-1	Q	a or ω_n	K	C_1	ω_c, $1/\epsilon$	R	ω_n/ω_c		C_2
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
filter order, n	ϵ_{dB}, 1, $\frac{f\text{-}3dB}{f\text{-}\epsilon dB}$	cutoff frequency	cosh a or 1	sinh a or 1	$\frac{\pi}{2n}$

Program Listing II

Step	Key	Comment
111	1/X	print design capacitance
112	PRTX	
113	SPC	stop program execution and
114	R/S	await operator decision
115	EEX	setup for next loop
116	STO0	
117	*LBL1	second order filter loop
118	SPC	
119	RCL0	calculate normalized
120	RCLI	pole locations:
121	x	
122	EEX	
123	→R	$\sigma_k = (\sinh a)(\sin((2k-1)\frac{\pi}{2n}))$
124	RCLD	
125	x	$\omega_k = (\cosh a)(\cos((2k-1)\frac{\pi}{2n}))$
126	X⇄Y	
127	RCLE	
128	x	
129	→P	calculate ω_{n_k} and Q_k, scale ω_{n_k} for proper normalized bandedge
130	RCL3	
131	x	
132	PRTX	
133	STO2	$\omega_{n_k} = \left[\omega_k^2 + \sigma_k^2\right]^{\frac{1}{2}} \cdot (K)$
134	X⇄Y	
135	COS	
136	ENT↑	$Q_k = \frac{1}{2\cos(\tan^{-1}\frac{\omega_k}{\sigma_k})}$
137	+	
138	1/X	
139	STO1	
140	PRTX	
141	2	increment 2k by 2
142	ST+0	
143	F0?	if even order filter, rtn
144	RTN	and await capacitor values
145	RCL0	odd order filter:
146	RCLA	jump if last section
147	X≤Y?	
148	GTO2	
149	RTN	await capacitor values
150	*LBLE	LOAD CAPACITOR VALUES
151	F2?	reject input if 3rd order
152	GTO3	section has been outputted
153	STO9	store C_2
154	X⇄Y	store C_1
155	STO4	
156	+	calculate and print R_1
157	RCL1	
158	x	$R_1 = \frac{\omega_n/\omega_c}{Q(C_1 + C_2)}$
159	RCL2	
160	RCL5	
161	÷	
162	STO7	
163	X⇄Y	
164	÷	
165	PRTX	
166	RCL4	calculate and print R_2
167	x	
168	RCL9	
169	x	
170	RCL7	$R_2 = \frac{(\omega_n/\omega_c)^2}{R_1 \cdot C_1 \cdot C_2}$
171	X²	
172	X⇄Y	
173	÷	
174	PRTX	
175	RCL0	test for loop exit
176	RCLA	
177	X>Y?	
178	GTO1	
179	SPC	loop exit
180	SPC	
181	RTN	
182	*LBL2	3rd order filter section
183	RCLE	calculate and print real 3rd order pole location
184	RCL3	
185	x	
186	PRTX	
187	SPC	
188	*LBL3	wait loop for second card read
189	SF2	
190	PSE	
191	GTO3	
192	*LBL4	print and set flag 3
193	PRTX	
194	SF3	
195	RTN	
196	*LBL5	entry subroutine
197	R↓	recover and store n
198	STOA	
199	2	
200	÷	
201	FRC	set flag 0 if n is even
202	CF0	
203	X=0?	
204	SF0	
205	Pi	
206	RCLA	calculate and store:
207	ENT↑	
208	+	$\frac{\pi}{2n} \rightarrow RI$
209	÷	
210	STOI	
211	EEX	
212	STO3	ω_n initialization
213	RTN	
214	*LBL6	exit routine,
215	SPC	clear flag 3 and space
216	CF3	
217	RTN	

NOTE TRIG MODE

LABELS

A	B	C	D	E
Buttr load n	Buttr load εdB	LOAD R	LOADf-3dB & START	LOAD CAPACITORS
a: Cheb load n↑dB	b:	c:	d: LOADf-εdB & START	e:
0: 2πf calc	1: 2nd order filter loop	2: 3rd order filter lp	3:	4: print & set flag 3
5: entry subroutine	6: exit subroutine	7:	8:	9:

FLAGS

Flag	Meaning
0	n even
1	Buttr
2	go to wait loop
3	data in

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	DEG	FIX
1		■	GRAD	SCI
2		■	RAD ■	ENG ■
3		■		n 3

Program Listing I

Step	Key	Comments
001	R/S	cancel pause after card read
002	*LBLE	LOAD O1↑O2↑O3 and START
003	SPC	
004	GSB9	test for P⇄S
005	P⇄S	execute and signal P⇄S
006	SF2	
007	STO3	store O3
008	R↓	store O2
009	STO2	
010	R↓	store O1
011	STO1	
012	RCL2	calculate and store:
013	x	$O_1 \cdot \Sigma OO \rightarrow R7$
014	RCL2	
015	RCL3	
016	x	
017	+	
018	RCL3	
019	RCL1	
020	x	
021	+	
022	RCL1	
023	x	
024	STO7	
025	GSB9	P⇄S and reset flag 2
026	RCL5	obtain and store ω_o
027	STOI	
028	RCLE	calculate $1/\tau$
029	RCL3	
030	x	
031	RCL1	recall: Q
032	RCL2	ω_n
033	RCL6	R
034	P⇄S	execute and signal P⇄S
035	SF2	
036	STO0	store R
037	R↓	store ω_n
038	STO4	
039	R↓	form and store 1/Q
040	1/X	
041	STO6	
042	X⇄Y	store $1/\tau$
043	STO5	
044	x	form $O_1(\frac{1}{\tau Q} + \omega_n)$
045	RCL4	
046	+	
047	RCL1	
048	x	
049	RCL2	form and store O2 + O3
050	RCL3	
051	+	
052	STOA	
053	x	form and store:
054	RCL4	$K_1 = \frac{\omega_n}{\omega_c^2}(\frac{1}{\tau Q} + \omega_n)O_1(O_2+O_3)$
055	x	
056	RCLI	
057	X^2	
058	÷	
059	STOB	
060	RCL4	form and store:
061	RCLI	$K_0 = (\frac{\omega_n^2}{\tau})(O2 + O3)(\frac{1}{\omega_c^3})$
062	÷	
063	X^2	
064	RCLI	
065	÷	
066	RCL5	
067	x	
068	RCLA	
069	x	
070	STOA	
071	RCL5	form and store:
072	RCL4	$K_2 = (\frac{1}{\tau} + \frac{\omega_n}{Q})(O_1 \Sigma OO)\frac{1}{\omega_c}$
073	RCL6	
074	x	
075	+	
076	RCL7	
077	x	
078	RCLI	
079	÷	
080	STOC	
081	RCL1	form and store:
082	RCL2	$K_3 = -(O_1 + O_2)(O_1 \Sigma OO)$
083	+	
084	RCL7	
085	x	
086	CHS	
087	STOD	
088	RCL0	form and store $10^{-8} \cdot R$ for iteration loop exit test
089	EEX	
090	8	
091	÷	
092	STO9	

REGISTERS									
0	1	2	3	4	5	6	7	8	9
S0 R1	S1 O1	S2 O2	S3 O3	S4 ω_n	S5 $1/\tau$	S6 1/Q	S7 $O1 \cdot \Sigma OO$	S8 f/f'	S9 $10^{-8} \cdot R$
A K_0	B $-K_1$	C K_2	D K_3	E sinh a or 1	I ω_c				

Program Listing II

Step	Key	Comments
093	*LBL0	Newton-Raphson loop for R1
094	RCL0	form and store:
095	RCL0	$f(R_1) = K_3R_1^3 + K_2R_1^2 + K_1R_1 + K_0$
096	RCL0	
097	RCLD	
098	x	
099	RCLC	
100	+	
101	x	
102	RCLB	
103	−	
104	x	
105	RCLA	
106	+	
107	STO8	
108	CLX	form:
109	+	$f'(R_1) = 3K_3R_1^2 + 2K_2R_1 + K_1$
110	+	
111	+	
112	RCLD	
113	x	
114	RCLC	
115	ENT↑	
116	+	
117	+	
118	x	
119	RCLB	
120	−	
121	ST÷8	form $-\Delta R_1 = f(R_1)/f'(R_1)$
122	RCL8	form $R_{1_{n+1}} = R_{1_n} + \Delta R_1$
123	ST−0	
124	ABS	iterate again if $\lvert\Delta R_1\rvert \geq 10^{-6} \cdot R_1$
125	RCL9	
126	X≤Y?	
127	GTO0	
128	RCL0	recall and print R_1
129	PRTX	
130	RCL5	calculate and print R_2
131	RCL4	$R_2 = \frac{\frac{1}{\omega_c}(\frac{1}{\tau} + \frac{\omega_n}{Q}) - R_1(O_1+O_2)}{O_2 + O_3}$
132	RCL6	
133	x	
134	+	
135	RCLI	
136	÷	
137	RCL1	
138	RCL2	
139	+	
140	RCL0	
141	x	
142	−	
143	RCL2	
144	RCL3	
145	+	
146	÷	
147	PRTX	
148	RCL0	calculate and print R_3
149	x	$R_3 = (\frac{1}{\omega_c^3})\frac{\omega_n^2/\tau}{R_1R_2O_1O_2O_3}$
150	RCL1	
151	x	
152	RCL2	
153	x	
154	RCL3	
155	x	
156	1/X	
157	RCL4	
158	RCLI	
159	÷	
160	X^2	
161	x	
162	RCLI	
163	÷	
164	RCL5	
165	x	
166	PRTX	
167	SPC	
168	*LBL9	if flag 2, execute P⇄S
169	F2?	
170	P⇄S	
171	RTN	

LABELS					FLAGS
A	B	C	D	E load capacitors	0
a	b	c	d	e	1
0 Newton-Raphson	1	2	3	4	2 P⇄S
5	6	7	8	9 P⇄S	3

SET STATUS			
FLAGS	ON	OFF	
0			
1			
2		■	
3			
TRIG	DEG	GRAD	RAD
DISP	FIX	SCI	ENG ■
n	3		

PROGRAM 2-11 DELIYANNIS POSITIVE AND NEGATIVE FEEDBACK ACTIVE RESONATOR DESIGN (USED FOR ACTIVE BANDPASS FILTERS).

Program Description and Equations Used

Active filter resonators are constrained by component value ranges (10 ohms to 10 megohms, 100 pF to 10 μF), operational amplifier gain-bandwidth limitations, and overall circuit sensitivities. The Deliyannis resonator circuit allows high Q realizations and also compensates for the finite gain and bandwidth of the operational amplifier [20].

This resonator synthesizes a second order pole pair of given ω_n and Q. The natural frequency, ω_n, and the quality factor, Q, are provided as outputs from the active Butterworth and Chebyshev filter programs contained in this section.

This resonator type has the ability to synthesize a resonator with infinite Q. The infinite Q resonator is used in the interior stages of the Szentirmai leapfrog filter topology [48]. The leapfrog active filter is a direct simulation of a passive LC filter, and generally has the same low sensitivity characteristics of the LC topology. When narrow-band active filters are required, the leapfrog topology will be one of the viable candidates for filter realization (also see the GIC realization in Program 2-6).

The circuit for the Deliyannis second order bandpass circuit is shown in Fig. 2-11.1.

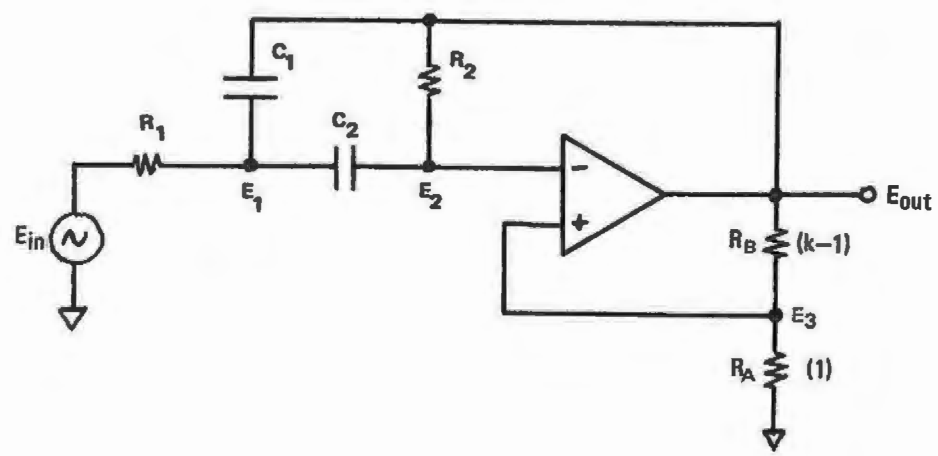


Figure 2-11.1 Deliyannis bandpass resonator circuit.

The transmission function is obtained using nodal analysis. In matrix form, the nodal equations are:

$$E_{out} = A(s)[E_3 - E_2], \text{ (op-amp transmission fcn)} \tag{2-11.1}$$

$$\begin{bmatrix} \{\frac{1}{R_1} + s(C_1 + C_2)\} & \{-sC_2\} \\ \{-sC_2\} & \{\frac{1}{R_2} + sC_2\} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} & sC_1 \\ 0 & \frac{1}{R_2} \end{bmatrix} \cdot \begin{bmatrix} E_{in} \\ E_{out} \end{bmatrix} \tag{2-11.2}$$

where

$$E_3 = E_{out}/k \tag{2-11.3}$$

Solving for E_2 from Eqs. (2-11.1) and (2-11.3):

$$E_2 = E_{out}\left[\frac{1}{k} - \frac{1}{A(s)}\right] \tag{2-11.4}$$

The transmission function is first obtained for the general case using A(s), then more specifically using $A(s) = A_o/(\tau s)$. The passive sensitivities may be obtained from the general solution, and the active sensitivities obtained from the specific solution, $A(s)=A_o/(\tau s)$.

The matrix equation is rewritten to bring 1/k − 1/A(s) inside the coefficient matrix, and to bring all dependent variables to the right

hand side of the equation:

$$\begin{bmatrix} \left\{\frac{1}{R_1} + s(C_1 + C_2)\right\} & \left\{-s\left[C_1 - C_2\left(\frac{1}{k} - \frac{1}{A(s)}\right)\right]\right\} \\ \left\{-sC\right\} & \left\{\left(\frac{1}{R_2}\right)\left(\frac{1}{k} - \frac{1}{A(s)} - 1\right) + sC_2\left(\frac{1}{k} - \frac{1}{A(s)}\right)\right\} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_{out} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} \\ 0 \end{bmatrix} \cdot E_{in} \tag{2-11.5}$$

Cramer's rule is used to find the expression for E_{out}/E_{in}, the filter transmission function.

$$\frac{E_{out}}{E_{in}} = \frac{-\dfrac{s}{R_1C_1\left(1 + {}^1/A(s) - {}^1/k\right)}}{s^2 + s\left\{\dfrac{1}{R_2C_1} + \dfrac{1}{R_2C_2} + \dfrac{1}{R_1C_1} \cdot \dfrac{{}^1/A(s) - {}^1/k}{1 + {}^1/A(s) - {}^1/k}\right\} + \dfrac{1}{R_1R_2C_1C_2}} \tag{2-11.6}$$

The passive sensitivities may be evaluated assuming the op-amp to be ideal, i.e., the open loop gain is allowed to approach infinity. In this situation, the transmission function becomes:

$$\frac{E_{out}}{E_{in}} = \frac{-\dfrac{ks}{R_1C_1(k-1)}}{s^2 + s\left\{\dfrac{1}{R_2C_1} + \dfrac{1}{R_2C_2} - \dfrac{1}{(k-1)R_1C_1}\right\} + \dfrac{1}{R_1R_2C_1C_2}} \tag{2-11.7}$$

The coefficients of the denominator of this equation may be compared with the like coefficients in the standard second order form to derive expressions for ω_n and Q. The standard second order form of the transmission function is:

$$\frac{E_{out}}{E_{in}} = \frac{ks}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2} \tag{2-11.8}$$

The following expressions for ω_n, Q, and K are obtained:

$$K = -\frac{k}{R_1 C_1 (k-1)} \tag{2-11.9}$$

$$\omega_n = (R_1 R_2 C_1 C_2)^{-\frac{1}{2}} \tag{2-11.10}$$

$$Q = \frac{\sqrt{\frac{R_2}{R_1}}}{\sqrt{\frac{C_2}{C_1}} + \sqrt{\frac{C_1}{C_2}} - \frac{1}{k-1} \cdot \frac{R_2}{R_1} \cdot \sqrt{\frac{C_2}{C_1}}} \tag{2-11.11}$$

Let $\mu = R_2/R_1$, and $\delta = C_2/C_1$, then:

$$Q = \frac{\sqrt{\mu\delta}}{\delta + 1 - \mu\delta/(k-1)} \tag{2-11.12}$$

The denominator of Eq. (2-11.12) can be made arbitrarily small by proper choice of μ. The denominator can be made to vanish completely causing Q to become infinite, thus generating the infinite Q resonator required for the interior stages of the leapfrog filter topology.

Sensitivities are a way of expressing how much a given parameter, say Q, is affected by a change in one of the circuit elements. The general convention is to express sensitivities as a demensionless number formed from the ratio of individual percentage changes:

$$S_R^Q = \lim_{\Delta R \to 0} \frac{\Delta Q/Q}{\Delta R/R} = \frac{R}{Q} \cdot \frac{\partial Q}{\partial R} \tag{2-11.13}$$

Applying this definition to the expressions for μ_n, Q, and K, the following passive sensitivities result:

$$S^{\omega_n}_{R_1,\,R_2,\,C_1,\,C_2} = -\tfrac{1}{2} \tag{2-11.14}$$

$$S^Q_{R_1} = -S^Q_{R_2} = -\tfrac{1}{2} - Q\,\frac{\sqrt{\mu\delta}}{k-1} \tag{2-11.15}$$

$$S^Q_{C_1} = -S^Q_{C_2} = -\tfrac{1}{2} + Q\sqrt{\mu\delta}\left(\frac{1}{\mu} - \frac{1}{k-1}\right) \tag{2-11.16}$$

$$S^Q_{R_B} = -S^Q_{R_A} = Q\,\frac{\sqrt{\mu\delta}}{k-1} \tag{2-11.17}$$

$$S^K_{R_1,\,C_2} = -1 \tag{2-11.18}$$

$$S^K_{R_A} = -S^K_{R_B} = {}^1/_k \tag{2-11.19}$$

The break frequency of the open loop transmission function of most operational amplifiers is around 10 Hz, and the gain-bandwidth product (GBP) is about 10^6 Hz thus, the finite gain characteristics of the op-amp begin to affect the active filter response when kilohertz frequencies are involved. In this frequency range, the operational amplifier transmission function, $A(s) = A_o/(1 + \tau s)$, may be approximated by $A(s) = A_o/\tau s$. With this approximation, the active filter transmission function becomes:

$$\frac{E_{out}}{E_{in}} = \frac{-\dfrac{s}{(R_1C_1)(1 + \tau s/A_o - 1/k)}}{s^2 + s\left\{\dfrac{1}{R_2C_1} + \dfrac{1}{R_2C_2} + \dfrac{1}{R_1C_1} \cdot \dfrac{\tau s/A_o - 1/k}{1 + \tau s/A_o - 1/k}\right\} + \dfrac{1}{R_1R_2C_1C_2}} \tag{2-11.20}$$

This expression is expanded, and like powers of s collected to form the final expression for the active filter transmission function:

$$\frac{E_{out}}{E_{in}} = \frac{s\,\dfrac{-k}{R_1C_1}}{D(s)} \tag{2-11.21}$$

where

$$D(s) = s^3\frac{k\tau}{A_o} + s^2\left\{k-1 + \frac{k\tau}{A_o}\left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_1}\right)\right\}$$

$$+ s\left\{(k-1)\left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right) - \frac{1}{R_1C_1} + \frac{\tau k}{A_oR_1R_2C_1C_2}\right\} + \frac{k-1}{R_1R_2C_1C_2}$$

The denominator is factored into a single pole and a complex conjugate pair:

$$\frac{E_{out}}{E_{in}} = \frac{H \cdot \dfrac{s}{\omega_n Q}}{\left(\dfrac{s}{\sigma} + 1\right)\left(\dfrac{s^2}{\omega_n^2} + \dfrac{s}{\omega_n Q} + 1\right)} \tag{2-11.22}$$

The natural frequency, ω_n, and the quality factor, Q, are derived by equating like powers of s between Eqs. (2-11.21) and (2-11.22):

$$\omega_n = \frac{1}{R_1 R_2 C_1 C_2} \cdot \sqrt{1 - \frac{\tau k^2}{A_o(k-1)\, R_1 C_1}} \qquad (2\text{-}11.23)$$

$$BW = \frac{\omega_n}{Q} = \left\{\frac{1}{R_2 C_1} + \frac{1^*}{R_2 C_2} + \frac{1}{R_1 C_1 (k-1)}\right\}\left\{1 - \frac{\tau k^2}{A_o (k-1) R_1 R_2}\right\} \qquad (2\text{-}11.24)$$

From these equations, the active sensitivities are derived:

$$S^{\omega_n}_{A/\tau} = S^{Q}_{A/\tau} = \tfrac{1}{2} \cdot \frac{\omega_n \tau}{A_o} \cdot \left(\frac{k}{k-1}\right)^2 \cdot \sqrt{\mu\delta} \qquad (2\text{-}11.25)$$

where $\mu = R_2/R_1$ (2-11.26)

and $\delta = C_2/C_1$ (2-11.27)

as defined previously. The objective is to choose μ or δ to strike a happy medium between the active and the passive sensitivities (see [19], p . 319).

The Designers Guide to Active Filters [26], has the set of equations that generate the element values for this positive and negative feedback biquad. The point is made that by choosing $\delta < 1$ some of the active sensitivities may be reduced at the expense of resistor value spread (μ increases).

Equations (2-11.28) through (2-11.42) are used by the HP-67/97 program. The equation solution starts with a choice for the capacitor ratio, δ, and positive feedback ratio, k, and the operational amplifier dc gain, A_o, and gain bandwidth product, GBP. The resonant frequency is f_o and $p = 1/k$ ($f_a = 1/(2\pi\tau)$).

$$\Omega = f_a/f_o = GBP/(f_o A_o) \qquad (2\text{-}11.28)$$

$$\gamma = A_o\,\Omega = GBP/f_o \qquad (2\text{-}11.29)$$

$$d = 1/Q \qquad (2\text{-}11.30)$$

$$\beta = \Omega - p\gamma = (1 - \frac{A_o}{k}) \qquad (2\text{-}11.31)$$

$$m = \gamma + \beta = \Omega\left\{1 + A_o\,(1 - \tfrac{1}{k})\right\} \qquad (2\text{-}11.32)$$

$$a_2 = (\delta + 1)\,\{\, m(m-d) + 1 \,\} \qquad (2\text{-}11.33)$$

$$a_1 = \delta m - (\delta + 1)\,\beta - (m-d)(md-1) \qquad (2\text{-}11.34)$$

$$a_0 = m\,(\beta - d) + 1 \qquad (2\text{-}11.35)$$

$$\left.\begin{aligned} C_1 &= 1 \\ C_2 &= \delta \end{aligned}\right\} \text{normalized values} \qquad (2\text{-}11.36)$$

The quadratic equation is used to find the positive real root (R_1) of:

$$a_2 R_1^{\,2} + a_1 R_1 + a_0 = 0 \qquad (2\text{-}11.38)$$

i.e.,

$$R_1 = \frac{-a_1}{2a_2} + \sqrt{\left(\frac{a_1}{2a_2}\right)^2 - \frac{a_0}{a_2}} \qquad (2\text{-}11.39)$$

then

$$R_2 = \frac{m(\delta + 1)\,R_1 - (dm - 1)}{(R_1 - \beta)\,\delta} \qquad (2\text{-}11.40)$$

H is the gain of the filter at resonance:

$$H = \frac{-R_2 \cdot \delta \cdot Q}{1 - \frac{1}{k} + \frac{1}{A_o}} \qquad (2\text{-}11.41)$$

A parasitic pole also exists. The location of this pole is at $-\sigma$, where:

$$\sigma = \frac{m}{\delta R_1 R_2} \qquad (2\text{-}11.42)$$

The normalized transmission function with the above element values becomes:

$$G(s) = \frac{E_{out}}{E_{in}} = \frac{\frac{H}{Q}\, s}{\left(s^2 + \frac{s}{Q} + 1\right)\left(\frac{s}{\sigma} + 1\right)} \qquad (2\text{-}11.43)$$

The design of this filter type is somewhat cut and try if low sensitivities are to be achieved. The program is written to take the desired resonant frequency, the operational amplifier parameters, the capacitor ratio, one capacitor value, and the positive feedback ratio, and provides the remaining element values.

Because the resonator design exhibits a gain, H, at resonance, the input resistor, R_1, may be split into two resistors to provide a Thevenin equivalent circuit with gain $H_{desired}/H = 1/H'$ and impedance R_1. This equivalent circuit is shown in Fig. 2-11.2.

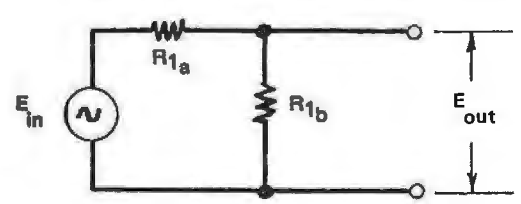


Figure 2-11.2 Equivalent input resistor network.

$$E_{out}/E_{in} = 1/H' = R_{1b}/(R_{1a} + R_{1b}) \tag{2-11.44}$$

$$R_{equiv} = (R_{1a} \cdot R_{1b})/(R_{1a} + R_{1b}) = R_1 \tag{2-11.45}$$

Equation (2-11.44) is solved for $R_{1a} + R_{1b}$, and substituted into Eq. (2-11.45) to yield an expression for R_{1a}:

$$R_{1a} = H' \cdot R_1 \tag{2-11.46}$$

Substituting Eq. (2-11.46) into Eq. (2-11.44) yields an expression for R_{1b}:

$$R_{1b} = R_{1a}/(H'-1) \tag{2-11.47}$$

Equations (2-11.46) and (2-11.47) are used by the program to split the input resistor and provide the desired resonator gain at the resonant frequency.

2-11 **User Instructions**

DELIYANNIS POSITIVE AND NEGATIVE FEEDBACK RESONATOR				
resistance level	δ	$H_{desired}$		
Q	p	op-amp gain-BW	op-amp dc gain	load f_o & start

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Load Q, the quality factor	Q	[A]	
3	Load p, the positive feedback ratio (p = 1/k)	p	[B]	
4	Load R, the operating resistance level	R, Ω	[f] [A]	
5	Load δ, the ratio of C2 to C1 (Eq. (2-10.27))	δ	[f] [B]	
6	Load desired gain at resonance	$H_{desired}$	[f] [C]	
7	Load op-amp gain-bandwidth product	GBP, Hz	[C]	
8	Load op-amp dc gain	A_o	[D]	
9	Load resonant frequency desired and start	f_o, Hz		R_1 (normalized)
				R_2 (normalized)
	note:			H (normalized)
	Flag 3 is tested on all input routines			σ (normalized)
	to determine whether input or output			R_B (normalized)
	of the respective parameter is desired.			
	If an input key ("A" - "D" and "a" - "c")			R_1 (denormalized)
	is keyed without numeric entry, or			R_2 (denormalized)
	following the clear key (e), the			C_1 (denormalized)
	presently stored parameter will be			C_2 (denormalized)
	displayed.			
				R_{1a} (denormalized)
				R_{1b} (denormalized)
				$S^{\omega_n}_{R_1, R_2, C_1, C_2}$
				$S^Q_{R_B}, -S^Q_{R_A}$
				$S^Q_{R_1}, -S^Q_{R_2}$
				$S^Q_{C_1}, -S^Q_{C_2}$
10	Go back and change any parameters in any order, and rerun program. The center frequency need not be reloaded unless it is being changed.			$S^{\omega}_{A/\omega}, -S^Q_{A/\omega}$

Example 2-11.1

A second order Deliyannis resonator is to be designed using a type 741 operational amplifier. The operational amplifier characteristics and resonator specifications are:

Center frequency:	1000 Hz
Q:	100
gain at resonance:	1.0
capacitor ratio:	1.0
p, positive fdbk ratio:	0.04
resistance level:	10000 Ω
op-amp gain-bandwidth:	500000 Hz
op-amp dc gain:	100000

Find the element values and calculate the sensitivities for this design. Investigate the effect of different values of positive feedback on the component value spread and sensitivities. The HP-97 printout for this problem is shown on the next page, and the schematic is shown in Fig. 2-11.3.

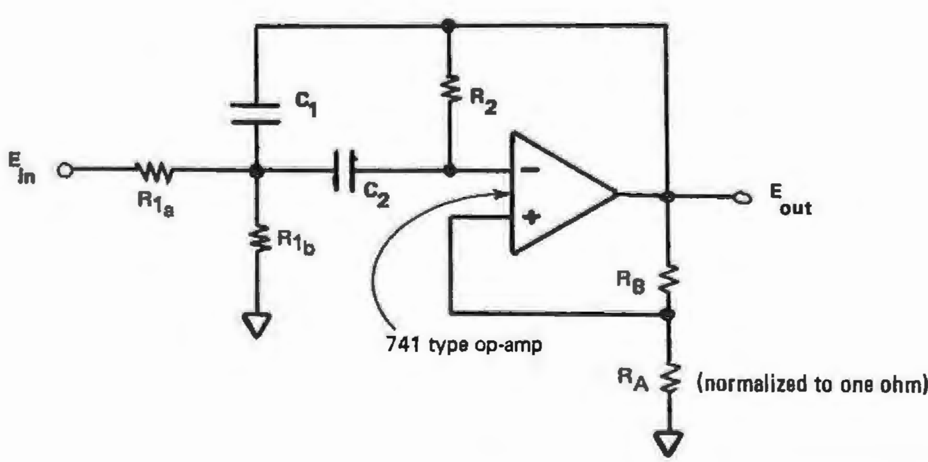


Figure 2-11.3 Deliyannis resonator schematic.

HP-97 printout for Example 2-11.1

100.	GSBA	**load Q**
10000.	GSBa	**load denormalization resistance level, R**
.04	GSBB	**load positive feedback ratio, p**
1.	GSBb	**load capacitor ratio, δ**
1.	GSBc	**load gain desired at resonance, $H_{desired}$**
500000.	GSBC	**load op-amp GBP**
100000.	GSBD	**load op-amp dc gain, A_o**
1000.	GSBE	**load f_o and start**

145.776-03	***	R_1	normalized values ($C_1 = 1$)
6.75947+00	***	R_2	
-704.104+00	***	H	
487.151+00	***	σ	
24.0000+00	***	R_B ($R_A = 1$)	
10.0000+03	***	R_1	denormalized values
463.707+03	***	R_2	
2.32001-09	***	C_1	
2.32001-09	***	C_2	
7.04104+06	***	R_{1A}	Thevenin equivalent input resistor pair
10.0142+03	***	R_{1B}	
-500.000-03	***	$S^{\omega_n}_{R_1, R_2, C_1, C_2}$	
28.3733+00	***	$S^Q_{R_B}, -S^Q_{R_A}$	
-28.8733+00	***	$S^Q_{R_1}, -S^Q_{R_2}$	
-14.1882+00	***	$S^Q_{C_1}, -S^Q_{C_2}$	
7.38889-03	***	$S^Q_{A/\tau}, -S^{\omega_n}_{A/\tau}$	

The following printouts have all parameters the same except the positive feedback ratio, p. Notice passive sensitivities increase and active decrease.

1.-09 GSBE **p**		.004 GSBE **p**		.4 GSBE **p**	
GSBE		GSBE		GSBE	
3.79141-03	***	46.3615-03	***	577.077-03	***
172.670+00	***	20.6707+00	***	1.71635+00	***
-17.2668+03	***	-2.07535+03	***	-286.053+00	***
763.761+00	***	519.661+00	***	302.893+00	***
1.00000+09	***	249.000+00	***	1.50000+00	***
10.0000+03	***	10.0000+03	***	10.0000+03	***
455.423+06	***	4.45860+06	***	29.7421+03	***
60.3422-12	***	737.866-12	***	9.18446-09	***
60.3422-12	***	737.866-12	***	9.18446-09	***
172.668+06	***	20.7535+06	***	2.86053+06	***
10.0006+03	***	10.0048+03	***	10.0351+03	***
-500.000-03	***	-500.000-03	***	-500.000-03	***
21.3406-06	***	8.48008+00	***	114.973+00	***
-500.021-03	***	-8.98008+00	***	-115.473+00	***
-31.4320-03	***	-4.24420+00	***	-57.4880+00	***
213.406-03	***	21.2853-03	***	4.79053-03	***

Program Listing I

Step	Key	Comments
001	*LBLA	LOAD Q
002	1/X	
003	STO0	store $d = 1/Q$
004	GTO0	
005	*LBLa	LOAD DENORMALIZATION
006	STO8	RESISTANCE LEVEL
007	GTO0	
008	*LBLB	LOAD p
009	STO1	
010	GTO0	
011	*LBLb	LOAD C_1/C_2 RATIO
012	STO2	
013	GTO0	
014	*LBLC	LOAD OP-AMP GBP
015	STO3	
016	GTO0	
017	*LBLc	LOAD $H_{desired}$
018	P⇄S	
019	STO0	
020	P⇄S	
021	GTO0	
022	*LBLD	LOAD OP-AMP A_o
023	STO4	
024	*LBL0	clear flag 3 subroutine
025	CF3	
026	RTN	
027	*LBLE	LOAD f_o AND START ANALYSIS
028	F3?	store f_o if entered
029	STO5	from keyboard
030	SPC	
031	RCL3	
032	RCL5	$\gamma = \frac{A_o}{f_o} \rightarrow R_A$
033	÷	
034	STOA	
035	RCL4	
036	÷	
037	RCLA	
038	RCL1	$\beta = \frac{\gamma}{A_o} - \frac{\gamma}{k} \rightarrow R_B$
039	x	
040	−	
041	STOB	
042	RCLA	
043	+	$m = \gamma + \beta$
044	STOC	
045	RCL2	
046	EEX	$\delta + 1 \rightarrow R9$
047	+	
048	STO9	
049	x	$m(\delta+1)$
050	RCLC	
051	RCL0	$m - d \rightarrow R_E$
052	−	
053	STOE	
054	x	
055	RCL9	$a_2 = (\delta+1)\{m(m-d)+1\} \rightarrow R_D$
056	+	
057	STOD	
058	RCL2	
059	RCLC	
060	x	
061	RCL9	$a_1 = \delta m - (\delta+1)\beta - (m-d)(dm-1)$
062	RCLB	
063	x	
064	−	
065	RCLC	
066	RCL0	
067	x	
068	EEX	$dm - 1 \rightarrow R_I$
069	−	
070	STOI	
071	RCLE	
072	x	
073	−	
074	2	$\frac{a_1}{2} \rightarrow R6$
075	÷	
076	STO6	
077	RCL0	
078	RCLB	
079	−	
080	RCLC	$a_o = m(\beta - d) + 1$
081	x	
082	EEX	
083	−	
084	RCLD	
085	ST÷6	
086	÷	
087	RCL6	
088	X²	
089	+	$R_1 = \sqrt{\left(\frac{a_1}{2a_2}\right)^2 - \frac{a_o}{a_2}} - \frac{a_1}{2a_2}$
090	√X	
091	RCL6	
092	−	
093	STOD	
094	PRTX	
095	RCLC	
096	RCL9	
097	x	
098	x	
099	RCLI	
100	−	
101	RCLD	
102	RCLB	$R_2 = \frac{m(\delta+1)R_1 - (dm-1)}{(R_1 - \beta)\delta}$
103	−	
104	÷	
105	RCL2	
106	÷	
107	STOE	
108	PRTX	
109	RCL2	
110	x	

REGISTERS

0	1	2	3	4	5	6	7	8	9
$d = \frac{1}{Q}$	$\rho = \frac{1}{k}$	$\delta = \frac{C_2}{C_1}$	op-amp GBP	op-amp dc gain, A_o	f_o	H, or a_1/a_2	$R_2\delta$, or μ	resistance level	$\delta+1$, or $Q\sqrt{\mu\delta}$
S0 $H_{desired}$	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
γ	β	m or $\epsilon = \frac{1}{k-1}$	a_2, or R_1	m−d, or R_2	dm−1, or −0.5

Program Listing II

Step	Key	Comments
111	STO7	
112	EEX	
113	RCL1	
114	−	
115	RCL4	
116	1/X	
117	+	$H_{actual} = \frac{-R_2 \cdot \delta \cdot Q}{1 - \frac{1}{k} + \frac{1}{A_o}}$
118	÷	
119	RCL0	
120	÷	
121	STO6	
122	CHS	
123	PRTX	
124	RCLC	
125	RCLD	
126	÷	
127	RCL7	$\sigma = \frac{m}{\delta \cdot R_1 \cdot R_2}$
128	÷	
129	PRTX	
130	RCL1	
131	1/X	
132	EEX	$R_B = k - 1$
133	−	
134	GSB9	
135	1/X	$\epsilon = \frac{1}{k-1}$
136	STOC	
137	RCL8	recall and print
138	PRTX	denormalized R_1
139	RCLE	calculate and print
140	x	denormalized R_2
141	RCLD	
142	÷	
143	PRTX	
144	RCL5	calculate and print
145	Pi	denormalized C_1:
146	x	
147	ENT↑	
148	+	
149	RCL8	$C_1 = \frac{R_1}{2\pi \cdot f_o \cdot R}$
150	x	
151	RCLD	
152	÷	
153	1/X	
154	PRTX	
155	RCL2	calculate and print
156	x	denormalized C_2
157	GSB9	
158	P⇄S	calculate Thevenin
159	RCL0	equivalent for R_1 to provide
160	P⇄S	desired gain at resonance:
161	ST÷6	
162	RCL6	
163	RCL8	
164	x	$R_{1a} = \frac{H_{actual}}{H_{desired}} \cdot R_1$
165	PRTX	
166	RCL6	
167	EEX	$R_{1b} = \frac{R_{1a}}{\frac{H_{actual}}{H_{desired}} - 1}$
168	−	
169	÷	
170	GSB9	
171	.	
172	5	$-0.5 \rightarrow R_I$
173	CHS	
174	STOI	
175	PRTX	print: $S^{\omega_n}_{C_1, C_2, R_1, R_2}$
176	RCLE	
177	RCLD	$\mu = \frac{R_2}{R_1} \rightarrow R_7$
178	÷	
179	STO7	
180	RCL2	
181	x	
182	√X	
183	RCL0	$S^Q_{R_B} = -S^Q_{R_A} = Q\frac{\sqrt{\mu\delta}}{k-1}$
184	÷	
185	STO9	
186	RCLC	
187	x	
188	PRTX	
189	CHS	
190	RCLI	$S^Q_{R_1} = -S^Q_{R_2} = -\frac{1}{2} - Q\frac{\sqrt{\mu\delta}}{k-1}$
191	+	
192	PRTX	
193	RCL7	
194	1/X	
195	RCLC	
196	−	
197	RCL9	$S^Q_{C_1} = -S^Q_{C_2} = -\frac{1}{2} + Q\sqrt{\mu\delta}\left(\frac{1}{\mu} - \frac{1}{k-1}\right)$
198	x	
199	RCLI	
200	+	
201	PRTX	
202	RCL9	
203	RCL0	
204	x	
205	RCLA	
206	ENT↑	
207	+	
208	÷	$S^{\omega}_{A/\gamma} = -S^{Q}_{A/\gamma} = \frac{\sqrt{\mu\delta}}{2} \cdot \frac{\omega_n \gamma}{A_o} \cdot \left(\frac{k}{k-1}\right)^2$
209	RCLC	
210	RCL1	
211	÷	
212	X²	
213	x	
214	*LBL9	print and space subroutine
215	PRTX	
216	SPC	
217	RTN	

LABELS

A	B	C	D	E
Load Q	Load p	Load GBP	Load A_o	Load f_o & Start
a Load R	b Load δ	c Load $H_{desired}$	d	e
0	1	2	3	4
5	6	7	8	9 Print & Space

FLAGS

Flag	Use
0	
1	
2	
3	Data entry

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
			USER'S CHOICE	
0		■	DEG	FIX
1			GRAD	SCI
2			RAD	ENG
3		■		n ____

PROGRAM 2-12 ELLIPTIC FILTER ORDER AND LOSS POLE LOCATIONS.

Program Description and Equations Used

This program finds the lowest elliptic (also called Cauer-Chebyshev) lowpass filter order that will meet the requirements for Amax, Amin, fmax, and fmin. These parameters are defined with the aid of Fig. 2-12.1.

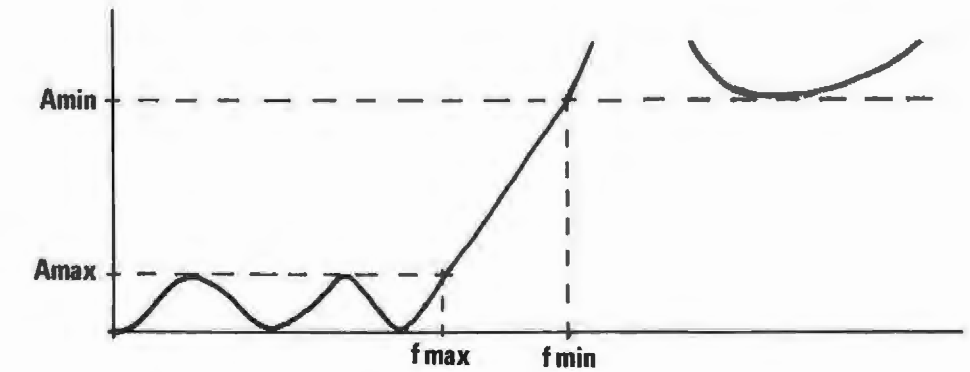


Figure 2-12.1 Elliptic filter loss function, where:

Amax : maximum passband ripple in dB
Amin : minimum stopband attenuation in dB
fmax : maximum passband frequency (passband edge)
fmin : minimum frequency where Amin is achieved.

The program also calculates the attenuation pole frequencies. From these frequencies the filter response at any frequency outside the passband may be determined by using the Z transformation. This transformation technique is described in the next program, and also in chapter 8 of Daniels' book [17]. The analog Z transformation should not be confused with the digital z transformation.

The elliptic filter response is not monotonic in the stopband as can be seen in Fig. 2-12.1. This stopband response is the characteristic difference between the Chebyshev and elliptic filter responses. Both filter types have equiripple behavior in the passband, but Chebyshev (and Butterworth) filters have all attenuation poles located at infinite frequency, while elliptic filters have finite attenuation poles. Because of these finite attenuation poles, the elliptic filter has a sharper transition from passband to stopband for a given filter order.

The elliptic response also has its drawbacks. As the transition band becomes sharper (the filter more selective) the transfer function phase angle changes more rapidly with frequency, and so the group delay becomes peaked near the passband edge frequency. Uniform group delay is required for filters that must process pulses without exhibiting ringing amplitude responses; thus, the transmission function of the elliptic filter tends toward the optimum only from the point of view of the attenuation requirement.

If the LC filter is being designed as a basis for an active filter design such as the leapfrog topology, or an elliptic response is being contemplated for active simulation by cascaded active resonators, the elliptic filter transmission zero (attenuation pole) simulation will require a biquadratic resonator circuit. The designer should always compare the sensitivities of the elliptic active filter circuit versus the sensitivities of a higher order all-pole active design which meets the overall same specifications. In general, as the active resonator circuit becomes more complicated, or the operating gain-bandwidth requirements approach the op-amp gain-bandwidth, the circuit sensitivities become worse, and the final filter design may not meet the specification requirements when component drift due to temperature and aging is considered.

The following formulas are discussed in detail in the equation derivation section and the results brought forward. The loss function, L, is defined by Eq. (2-12.1) (refer to Fig. 2-12.1).

$$L^2 = \frac{10^{0.1\,\text{Amin}} - 1}{10^{0.1\,\text{Amax}} - 1} \tag{2-12.1}$$

Furthermore, x_L is the ratio of the lowpass stopband edge frequency to the lowpass passband edge frequency (refer to Fig. 2-12.1):

$$x_L^{-1} = \frac{f_{max}}{f_{min}} \tag{2-12.2}$$

The minimum elliptic filter order that will meet the requirements for Amax, Amin, fmax, and fmin is calculated from Eq. (2-12.28).

$$n = \frac{K\left(x_L^{-1}\right) \cdot K'\left(L^{-1}\right)}{K'\left(x_L^{-1}\right) \cdot K\left(L^{-1}\right)} \tag{2-12.30}$$

where K( ) is the complete elliptic integral of the first kind, and K'( ) is the complementary complete elliptic integral of the first kind. These functions are defined by Eqs. (2-12.11) through (2-12.14) and are calculated by a truncated infinite series as given by Eqs. (2-12.18) through (2-12.21).

The loss poles of the elliptic filter transfer function are given by Eqs. (2-12.31) and (2-12.32).

$$x_\nu = \frac{x_L}{x_{z\nu}} \tag{2-12.31}$$

where

$$x_{z\nu} = \begin{cases} \text{sn}\left\{\frac{2\nu}{n} K\left(x_L^{-1}\right),\ x_L^{-1}\right\} & n \text{ odd} \\ \text{sn}\left\{\frac{2\nu-1}{n} K\left(x_L^{-1}\right),\ x_L^{-1}\right\} & n \text{ even} \end{cases} \tag{2-12.32}$$

The elliptic sine is evaluated by means of a Fourier series given by Eqs. (2-12.24) and (2-12.25).

The even ordered elliptic filters have a stopband loss that approaches a constant, finite value as the frequency approaches infinity, i.e., the even ordered elliptic filter does not have a loss pole at infinite frequency. The lossless LC synthesis of such a filter cannot be done without the use of mutual inductive coupling between the filter sections. On the other hand, active filter realizations can be done without the loss pole locations being a constraint.

A special form of the Möbius transformation (a bilinear change of variables) may be applied to the even ordered elliptic loss pole frequencies to move the highest frequency loss pole to infinity and thereby allow LC synthesis without mutual inductance. The even ordered elliptic filter element value tables in Zverev [58], already have this transformation applied, hence $x_L^{-1} = \sin\,\theta$ only for odd order filters (θ is the tabulated modular angle).

The general form of the Möbius transformation is:

$$s^2 = \left\{\frac{\Omega_C^{\,2} - \Omega_B^{\,2}}{\Omega_B^{\,2} - \Omega_0^{\,2}}\,\Omega_B^{\,2}\right\}\cdot\left\{\frac{s^2 + \Omega_0^{\,2}}{s^2 + \Omega_C^{\,2}}\right\} \qquad (2\text{-}12.3)$$

This transformation converts frequencies as follows:

1) $S = j\,\Omega_0$ to $s = 0$

2) $S = j\,\Omega_B$ to $s = j\Omega_B$ (no change in passband edge)

3) $S = j\,\Omega_C$ to $s = \infty$

It is not desired to transform the dc, or zero frequency, location in the lowpass filter, hence, $\Omega_0 = 0$; furthermore, the loss poles lie directly on the $j\omega$ axis so the transformation need only apply to $s = j\omega$, thus Eq. (2-12.3) becomes:

$$\omega^2 = \left(\Omega_C^{\,2} - \Omega_B^{\,2}\right)\frac{\Omega^2}{\Omega_C^2 - \Omega^2} \qquad (2\text{-}12.4)$$

The program calculates and prints (displays) the original even-ordered pole locations as calculated from Eq. (2-12.32) applies Eq. (2-12.4), and prints and stores the transformed pole locations. For odd-ordered filters, the program calculates, prints, and stores the finite loss pole locations from Eq. (2-12.32) without transformation. In both the even and odd cases, the loss pole frequencies are stored in normalized form ($\Omega = 1$), but are denormalized for printout or display.

The normalized loss pole frequencies are used by the next program in this section to calculate the filter attenuation at any frequency within the passband or the stopband by using the Z transform.

User Instructions

ELLIPTIC FILTER ORDER AND LOSS POLE LOCATIONS				
Amin	fmin	change n		
Amax	fmax	compute n	compute loss poles	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Load maximum passband ripple in dB	Amax	[A]	
3	Load minimum stopband loss in dB	Amin	[f] [A]	
4	Load passband cutoff frequency	fmax	[B]	
5	Load minimum stopband loss frequency	fmin	[f] [B]	
6	Calculate filter order to meet requirements		[C]	n n*
	*The first n will be the result of the calculations and will generally not be an integer. The second n is the next highest integer, and is the stored value. Both values are given so the designer can get a feeling for the design margin. If the two values are close, the next higher filter order might be considered. If the program stops displaying "Error", the input data for Amax and Amin are too far apart (.005dB and 100dB for example) and calculations for $K(L^{-1})$ exceed the precision capability of the HP-97. The filter order may be obtained from the Kawakami CC nomograph [34], [58], and the program restarted with step 7. Step 8 will still run correctly.			
7	To change filter order (integers only)	n	[f] [C]	
8	To calculate loss poles (frequencies of maximum attenuation)		[D]	f_1 f_2 ⋮ $f_{n/2}$** space f_1' *** f_2' ⋮ $f'_{(n-1)/2}$
	**The number of loss poles will be the integral part of n/2, i.e., a fifth order filter will have two loss poles. ***If n is even, the Möbius transformation is done to ensure a loss pole at infinity. The primed frequencies (f') are the Möbius transformed frequencies. The highest original loss frequency has been transformed to infinite frequency, and is not printed out, i.e., a sixth order filter only has two transformed loss frequencies printed out. The original frequency, $f_{n/2}$, is the transformed fmin frequency.			

Example 2-12.1

Compute the filter order and loss pole locations for an elliptic filter to meet the following specifications.

A_{max} = .28 dB (ρ= 25%, $A_{max} = -10 \log(1-\rho^2)$)

A_{min} = 63 dB

f_{max} = 1000 Hz

f_{min} = 2000 Hz

HP-97 input/output

```
        .28 GSBA  load Amax
      63.00 GSBa  load Amin
    1000.00 GSBE  load fmax
    2000.00 GSBb  load fmin
            GSBC  calculate minimum filter order

       4.97  ***  actual calculated filter order, n
       5.00  ***  nearest integral value for n to meet specs

            GSBD  calculate loss pole locations
3250.804880  ***
2089.246505  ***
```

These results may be checked by comparing them to the 30° modular angle filter design shown in the "Catalog of Normalized Lowpass Models" on page 220 of Zverev [58].

Example 2-12.2

Compute the minimum filter order and loss pole locations for an elliptic filter which meets the following specifications:

A_{max} = .1773 dB (ρ= 20%, $A_{max} = -10 \log(1-\rho^2)$)

A_{min} = 78 dB

f_{max} = 1000 Hz

f_{min} = 2000 Hz

```
      .1773 GSBA  load Amax
      78.00 GSBa  load Amin
    1000.00 GSBE  load fmax
    2000.00 GSBb  load fmin
            GSBC  calculate minimum filter order

       5.96  ***  actual calculated filter order, n
       6.00  ***  nearest integral n to meet specs

            GSBD  calculate loss pole locations:
7235.802719  ***  }
2732.053611  ***  }   untransformed loss poles
2061.185330  ***  }
      (2061.185330) <-- also represents transformed fmin

2922.132266  ***  }
2129.548771  ***  }   transformed loss pole locations
```

Derivation of Equations Used

The elliptic response is governed by the Chebyshev rational function, which is a ratio of polynomials. The development of the Chebyshev rational function in terms of elliptic functions is beyond the scope of this discussion. This development is discussed in Chapter 5 of Daniels' book [17]. A few highlights of the Chebyshev rational function and elliptic functions will be used to show the development of the equations used by this program.

The Chebyshev response becomes the elliptic response when the Chebyshev polynomial, $T_n(x)$, is replaced by the Chebyshev rational function, $R_n(x,L)$, in the filter transfer function (Feldtkeller equation).

$$|H(j\omega)|^2 = 1 + |K(j\omega)|^2 \qquad (2\text{-}12.5)$$

for Chebyshev response, $|K(j\omega)|^2 = \varepsilon^2 \cdot T_n\ (x)$ (2-12.6)

for elliptic response, $|K(j\omega)|^2 = \varepsilon^2 \cdot R_n\ (x,L)$ (2-12.7)

Hence, the elliptic attenuation function is:

$$A(\omega)_{dB} = 20 \cdot \log|H(j\omega)| \quad (2\text{-}12.8)$$

$$= 10 \cdot \log\left\{1 + \varepsilon^2 \cdot R_n^{\ 2}(x,L)\right\} ;$$

$$\text{where } x = \omega/\omega max = f/fmax \quad (2\text{-}12.9)$$

The Chebyshev rational function, $R_n(x,L)$, has the following properties (also see Fig. 2-11.2).

1) R_n is odd when n is odd and vice versa.

2) All the zeros of R_n lie within the interval $-1 < x < 1$, while all the poles lie outside this interval.

3) $R_n(x,L)$, like $T_n(x)$, oscillates between ± 1 for $-1 < x < 1$. This interval defines the passband.

4) $R_n(1,L) = +1$ (passband edge).

5) $|R_n| > L$ (oscillates outside of L) for $|x| > x_L$, where x_L is defined as the first value of x where $R_n(x,L) = L$, and hence, Amin is achieved (defines stopband).

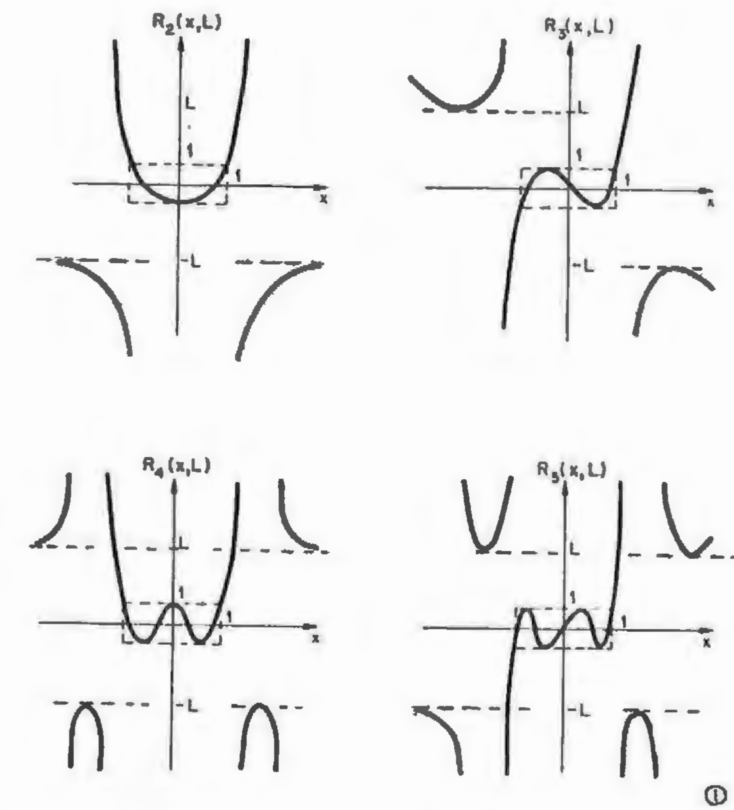


Figure 2-11.2 Chebyshev rational functions for n = 2 to 5.

By using Eq. (2-12.8) and condition 5, an expression for L can be found in terms of the filter parameters Amin and ε.

$$L^2 = \frac{10^{0.1Amin} - 1}{\varepsilon^2} \quad (2\text{-}12.10)$$

Since $A(\omega) = Amax$ at the passband edge, fmax, condition 4 and Eq. (2-12.8) can be used to find an expression for ε.

$$\varepsilon^2 = 10^{0.1Amax} - 1 \quad (2\text{-}12.11)$$

Not surprisingly, this is the same expression as is used in the Chebyshev case, and for the same reasons (condition 3).

By putting Eqs. (2-12.10) and (2-12.11) together, the expression for L is obtained:

$$L^2 = \frac{10^{0.1Amin} - 1}{10^{0.1Amax} - 1} \quad (2\text{-}12.12)$$

ELLIPTIC FUNCTIONS

There are three kinds of elliptic integrals (see Abramowitz and Stegun, [1]). Only the elliptic integral of the first kind is needed for elliptic filters. The elliptic integral of the first kind is defined by the following equation:

$$u(\phi,k) = \int_0^{\phi} \frac{dx}{(1 - k^2 \cdot \sin^2 x)^{1/2}} \qquad (2\text{-}12.13)$$

The two variables, ϕ and k, are called the amplitude and modulus respectively. Some elliptic function tables [1], and some elliptic filter tables [58], are parametric in terms of the modular angle, θ, instead of the modulus, k. The modular angle is defined by:

$$k = \sin\theta \qquad (2\text{-}12.14)$$

The complete elliptic integral of the first kind results when ϕ, the limit of integration, is taken as $\pi/2$ radians. This value, $u(\pi/2,k)$ is defined as $K(k)$.

Figure 2-12.3 shows $u(\phi,k)$ parametric with the modular angle, θ. $u(\phi,k)$ has been normalized with respect to $K(k)$. Figure 2-12.4 shows the complete elliptic integral, $K(k)$ by itself.

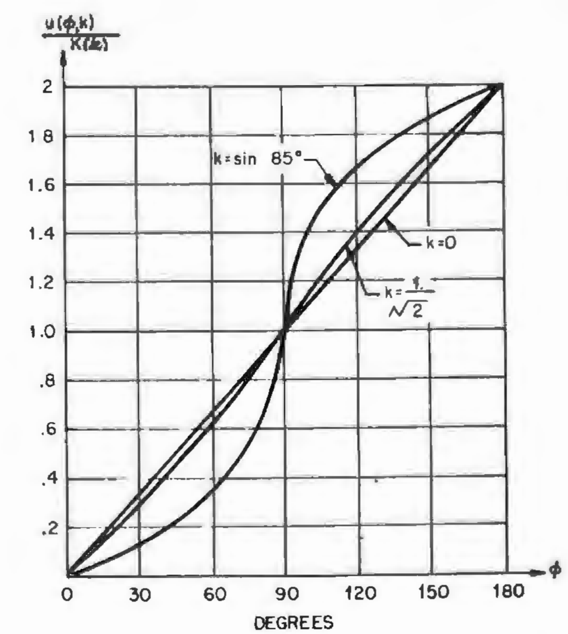


Figure 2-12.3 Elliptic integral.

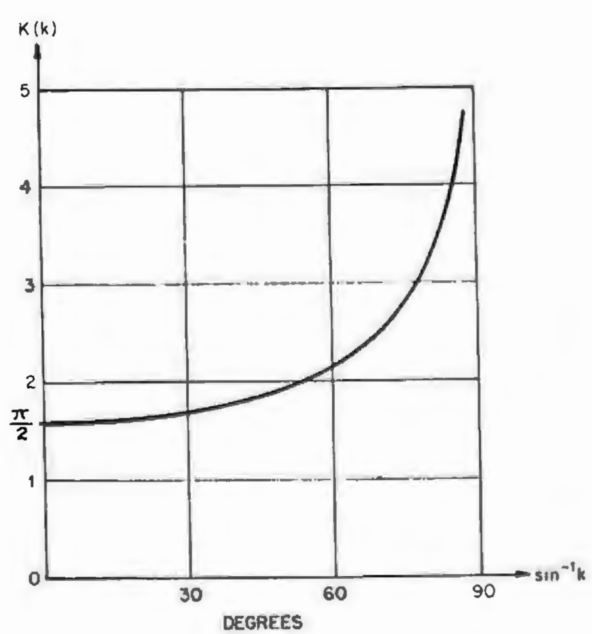


Figure 2-12.4 Complete elliptic integral.

The complementary modulus is defined in terms of the modulus, k, or the modular angle, θ, as:

$$k' = (1 - k^2)^{1/2} = \cos\theta \qquad (2\text{-}12.13)$$

The complementary complete elliptic integral is defined in terms of the complementary modulus:

$$K'(k) = K(k') = u(\pi/2, k') \qquad (2\text{-}12.16)$$

The elliptic sine is an elliptic function, and is defined in a somewhat reverse manner from the elliptic integral:

$$u(\phi,k) = \int_0^{\phi} (1 - k^2 \cdot \sin^2(x))^{-1/2}\, dx \qquad (2\text{-}12.17)$$

$$sn(u,k) = \sin\phi \quad \text{(elliptic sine)} \qquad (2\text{-}12.18)$$

$$cn(u,k) = \phi \quad \text{(elliptic cosine)} \qquad (2\text{-}12.19)$$

The definition is "reverse" since the limit of integration, ϕ, must be found to yield the "input," $u(\phi,k)$ and k. Figure 2-12.5 shows the elliptic sine and elliptic cosine functions.

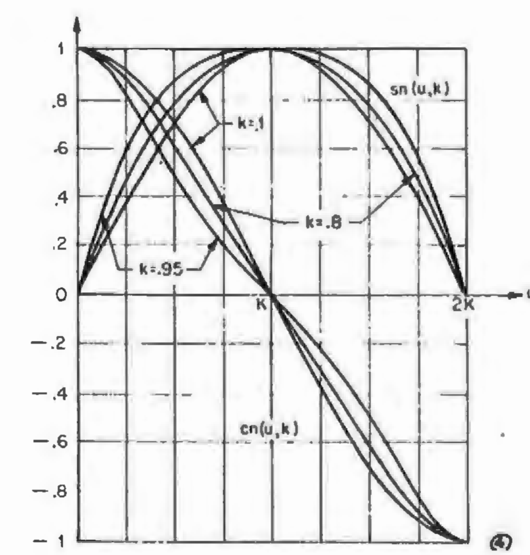


Figure 2-12.5 Elliptic sine and cosine functions.

Luckily, there are rapidly converging series expansions for both K(k) and sn(u,k), [12], and the programmable calculator can be used to perform the iterative calculations. These series expansions are:

Complete elliptic integral

$$K(k) = \frac{\pi}{2} \prod_{m=o}^{\infty} (1 + k_{m+1}) \; ; \qquad (2\text{-}12.20)$$

where

$$k_{m+1} = (1 = k_m')/(1 + k_m') \qquad (2\text{-}12.21)$$

$$k_m' = (1 = k_m^2)^{\frac{1}{2}} \text{, (complementary modulus)} \qquad (2\text{-}12.22)$$

$$k_o \equiv k \qquad (2\text{-}12.23)$$

The terms of the infinite product expansion rapidly converge toward unity. The series is terminated when $k_m < 10^{-9}$. This accuracy is generally achieved in four iterations or less.

Elliptic sine

The elliptic sine is calculated from the following Fourier series:

$$s_n(u,k) = \frac{2\pi}{K(k)\cdot k} \sum_{m=o}^{\infty} \left\{ \frac{q^{m+\frac{1}{2}}}{1 - q^{2m+1}} \right\} \cdot \sin\left((2m+1) \frac{\pi u}{2K(k)}\right) \qquad (2\text{-}12.24)$$

where q is Jacobi's nome (also called modular elliptic function):

$$q = e^{-\frac{\pi K'(k)}{K(k)}} \qquad (2\text{-}12.25)$$

The series is terminated when $(q^{m+\frac{1}{2}})/(1 - q^{2m+1}) < 10^{-9}q$

This particular algorithm for the elliptic sine is only one of many which can be used to calculate the function. For sharp cutoff filters, the convergence is slow; however, of all the algorithms researched by the author, the Fourier series method could be coded to fit into the HP-97 program memory and still leave enough room for the coding needed for the rest of the program.

If more registers were available, the descending Landen transformation method could have been combined with the calculation of K(k) to simultaneously yield K(k) and sn (u,k) as outlined in Skwirzynski and Zdunek's article [46]. If more program space were available, the elliptic sine could be calculated from the ratios of sums of hyperbolic sines and cosines as recommended by Orchard [41]. Also, if more program space were available, the calculation of the transmission zeros could be done directly from adaptations of the elliptic sine as represented by infinite products of hyperbolic tangents given by Amstutz [2] or as interpreted by Geffe [27]. Darlington's algorithm [18] is used in Program 2-15, and is a concise method for calculating the transmission zeros and poles when the filter order is odd.

Filter order calculation: Just as the trigonometric sine is periodic, so is the elliptic sine, although the elliptic sine is doubly periodic with a real period of $4\cdot K(k)$, and an imaginary period of $2\cdot K'(k)$. The Chebyshev rational function, R(x,L) may be expressed in terms of the complete elliptic integral and the elliptic sine. By relating the real and imaginary periods of the elliptic sine function to the real and imaginary periods of the Chebyshev rational function, two equations in two unknowns, C and n, may be formulated. These equations are:

Chebyshev rational function and elliptic functions

$$R_n(x,L) = \begin{cases} \text{sn}\left(uL/C,\ L^{-1}\right) & \text{n odd} \\ \text{sn}\left(uL/C + (-1)^{\frac{n}{2}} \cdot K(L^{-1}),\ L^{-1}\right) & \text{n even} \end{cases} \qquad (2\text{-}12.26)$$

where C is a constant, and u is the solution to:

$$x = \text{sn}(x_L \cdot u,\ x_L^{-1}) \qquad (2\text{-}12.27)$$

Simultaneous equations in C and n:

$$x_L^{-1} \cdot K(x_L^{-1}) = n \cdot C \cdot L^{-1} \cdot K(L^{-1}) \quad \text{(real periods)} \qquad (2\text{-}12.28)$$

$$x_L^{-1} \cdot K'(x_L^{-1}) = C \cdot L^{-1} \cdot K'(L^{-1}) \quad \text{(imaginary periods)} \qquad (2\text{-}12.29)$$

Eliminating C by simultaneous solution of Eqs. (2-12.28) and (2-12.29) results in the following expression for the filter order, n:

$$n = \frac{K(x_L^{-1}) \cdot K'(L^{-1})}{K'(x_L^{-1}) \cdot K(L^{-1})} \qquad (2\text{-}12.30)$$

where x_L^{-1} is defined by Eq.(2-12.2) and L^{-1} by Eq. (2-12.18):

$$x_L^{-1} = (fmax)/(fmin)$$

$$L^{-1} = \left\{ \frac{10^{0.1Amax} - 1}{10^{0.1Amin} - 1} \right\}^{1/2}$$

The loss poles of the elliptic filter transfer function, Eq. (2-12.5), are given by:

$$x_\nu = \frac{x_L}{x_{z\nu}} \qquad (2\text{-}12.31)$$

where:

$$x_{z\nu} = \begin{cases} sn\left(\frac{2\nu}{n} K(x_L^{-1}), x_L^{-1}\right) & n \text{ odd} \\ sn\left(\frac{(2\nu-1)}{n} K(x_1^{-1}), x_L^{-1}\right) & n \text{ even} \end{cases} \quad \nu = 1, 2, \ldots, n \qquad (2\text{-}12.32)$$

In Eq. (2-12.22) k becomes x_1^{-1} for the above elliptic sine computation, hence:

$$x_\nu = \frac{K(x_L^{-1})}{2\pi\Sigma} \qquad (2\text{-}12.33)$$

where Σ is the term summation in Eq.(2-12.24).

Program Listing I

Key	Comments
*LBLA	LOAD Amax (passband ripple)
STO0	
RTN	
*LBLa	LOAD Amin (min stopband loss)
STO1	
RTN	
*LBLB	LOAD fmax
STO2	
RTN	
*LBLb	LOAD fmin
STO3	
RTN	
*LBLC	calculate filter order, n
RCL2	compute and store:
RCL3	$x_L^{-1} = \frac{f_{max}}{f_{min}} \rightarrow R9$
÷	
STO9	
GSB5	compute and store:
STO4	$K(x_L^{-1}) \rightarrow R4 \rightarrow R7$
STO7	
RCL9	compute and store:
GSB4	$K'(x_L^{-1}) \rightarrow R8$
STO8	
ST÷4	continue n calculation
RCL0	compute and store:
GSB7	
RCL1	
GSB7	$L^{-1} = \sqrt{\frac{10^{0.1\,Amax} - 1}{10^{0.1\,Amin} - 1}} \rightarrow R_E$
÷	
√X	
STOE	
X²	generate error message if
RCLC	L^{-1} is smaller than 10^{-5}
X>Y?	
GTO9	call to unused label: "ERROR"
RCLE	compute: $K'(L^{-1})$
GSB4	
ST×4	continue n calculation
RCLE	compute: $K(L^{-1})$
GSB5	
ST÷4	finish n computation
RCL4	recall n
SPC	
PRTX	print non-integral n
EEX	convert n to next highest
STOD	integer, print and store
+	
INT	
PRTX	
*LBLc	LOAD ALTERNATE n VALUE
STO4	
GTO3	
*LBLD	CALCULATE LOSS POLES
DSP9	set display format
1	initialize index register
3	
STOI	
.	calculate starting ν :
5	n even, $\nu = \frac{1}{2}$
STO6	n odd, $\nu = 1$
RCL4	
×	
FRC	
ST+6	
SF0	set flag 0 if n is odd
X=0?	
CF0	
Pi	compute and store q:
RCL8	$q = e^{-\pi\frac{K'}{K}}$
×	
RCL7	
÷	
CHS	
e^x	
STOE	
RCLC	calculate error limit for loop exit
×	
STOC	
*LBL0	loop to calculate elliptic sine $(sn(x, x_L^{-1}))$
Pi	
RCL6	
×	compute and store: $(\pi\nu)/n = x$
RCL4	
÷	
P⇄S	
STO0	
EEX	initialize 2m + 1
STO1	
CLX	initialize summation (Σ)
STO3	
*LBL1	elliptic sine loop
RCLE	compute and store:
RCL1	
Y^x	
√X	$\frac{q^{m+\frac{1}{2}}}{1 - q^{2m+1}} \rightarrow S2$
LSTX	
CHS	
EEX	
+	
÷	
STO2	
RCL1	compute and add to sum (Σ):
RCL0	
×	
SIN	$(S2)\sin((2m+1)\pi\nu/n)$
×	
ST+3	
2	increment 2m
ST+1	
RCLC	test for loop exit:
RCL2	
X>Y?	

REGISTERS

0	1	2	3	4	5	6	7	8	9
Amax	Amin	fmax	fmin	n	scratch	index, ν	$K(x_L^{-1})$	$K'(x_L^{-1})$	x_L^{-1}

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
$(\pi\nu)/n$	2m + 1	$\frac{q^{m+1/2}}{1-q^{2m+1}}$	Σ	← loss pole storage registers →					

A	B	C	D	E	I
← loss pole storage →		10^{-10}	NINF (used by next program)	scratchpad	storage reg. index

2-12

Program Listing II

Step	Key	Comments
113	GTO1	
114	RCL3	recall summation & compute sn:
115	P⇄S	
116	ENT↑	$\frac{sn(x, x_L^{-1})}{x_L} = \frac{2\pi \sum}{K(x_L^{-1})}$
117	+	
118	Pi	
119	×	
120	RCL7	
121	÷	
122	1/X	compute and store normalized loss pole locations
123	ISZI	
124	STOi	
125	RCL2	denormalize and print loss pole locations
126	×	
127	PRTX	
128	EEX	increment register index
129	ST+6	
130	RCL6	test for loop exit:
131	ENT↑	
132	+	loop if $n > 2\nu$
133	RCL4	
134	X>Y?	
135	GTO0	
136	SPC	
137	F0?	jump if n is odd
138	GTO3	
139	1	initialize index register and store highest register number for later exit test
140	4	
141	X⇄I	
142	STOE	
143	RCLi	recall Ω_c
144	X²	store Ω_c^2
145	STO5	
146	*LBL2	Möbius transformation loop
147	ISZI	calculate Möbius transform for even ordered lowpass:
148	RCLi	
149	X²	
150	RCL5	
151	LSTX	
152	X²	
153	−	$\omega^2 = (\Omega_c^2 - 1)\frac{\Omega^2}{\Omega_c^2 - \Omega^2}$
154	÷	
155	RCL5	
156	EEX	
157	−	
158	×	
159	ABS	
160	√X	
161	DSZI	
162	STOi	store normalized & xfmed loss pole location
163	ISZI	
164	RCL2	denormalize and print loss pole location
165	×	
166	PRTX	
167	RCLI	
168	RCLE	
169	X≠Y?	test for loop exit
170	GTO2	
171	DSZI	restore highest reg. index
172	2	NINF = 2 for transformed even ordered filters
173	STOD	
174	*LBL3	
175	DSP2	set original display format
176	SPC	
177	RTN	return control to keyboard
178	*LBL4	compute K (k)
179	X²	
180	CHS	form complementary modulus:
181	EEX	$k' = \sqrt{1 - k^2}$
182	+	
183	√X	
184	*LBL5	compute K'(k)
185	STO5	store argument, k
186	Pi	initialize product register
187	2	
188	÷	
189	STO6	
190	*LBL6	complete elliptic integral
191	EEX	
192	RCL5	
193	X²	$K(x) = \frac{\pi}{2}\prod_{m=0}^{\infty}(1 + k_m)$
194	−	
195	√X	
196	EEX	
197	X⇄Y	$k_{m-1} = \frac{1 - k_m'}{1 + k_m'}$
198	−	
199	LSTX	
200	EEX	$k_i' = \sqrt{1 - k_i^2}$
201	+	
202	÷	
203	STO5	
204	EEX	
205	+	
206	ST×6	
207	RCL5	
208	EEX	
209	CHS	test for loop exit:
210	1	loop if $k_m > 10^{-10}$
211	0	
212	STOC	
213	X≤Y?	
214	GTO6	
215	RCL6	recall K(k)
216	RTN	return to main program
217	*LBL7	subroutine to compute:
218	EEX	$10^{0.1A} - 1$
219	1	
220	÷	
221	10ˣ	
222	EEX	
223	−	
224	RTN	

LABELS

A load Amax	B load fmax	C calc n	D calc loss poles	E
a load Amin	b load fmin	c enter n	d	e
0 sn(u,k)	1 sn loop start	2 Mobius transform	3 jump dest	4 K(k)
5 K'(k)	6 K'(k) loop	7 conversion subroutine	8	9

FLAGS

0	n odd
1	
2	
3	

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0		■	DEG		FIX	■
1			GRAD		SCI	
2			RAD	■	ENG	
3					n 2	

PROGRAM 2-13 RESPONSE OF A FILTER WITH CHEBYSHEV PASSBAND AND ARBITRARY STOPBAND LOSS POLES.

Program Description and Equations Used

This program will calculate the passband and stopband attenuation of lowpass, highpass, bandpass, and bandstop filters having Chebyshev (equi-ripple) passbands and arbitrary stopband losspole locations. The elliptic filter is a special case of this filter class in that the loss pole locations are chosen to provide equi-ripple stopband behavior.

Bandpass and bandstop filters are assumed to be the classic transformations of the lowpass structure, i.e., equal numbers of attenuation poles on either side of the passband, and geometrical symmetry of those poles about the center frequency. The program is designed to take either stored normalized lowpass loss pole frequencies provided by Program 2-12, or to accept normalized lowpass loss pole frequencies, number of poles at infinite frequency, and passband ripple as provided by the user.

This program is adapted from an unpublished HP-67/97 elliptic stopband attenuation program written by Philip R. Geffe. The basis of the program is the Z transformation, and the associated loss function, L(Z). This function allows the calculation of the stopband attenuation of equi-ripple passband elliptic filters from a knowledge of the loss pole frequencies only [17]. The transformed variable, Z, is defined by:

$$Z^2 = (s^2 + \omega_B^2)/(s^2 + \omega_A^2) \qquad (2\text{-}13.1)$$

This function spreads the passband ($s = j\omega_A$ to $j\omega_B$) over the entire imaginary Z axis, and spreads the stopbands along the real Z axis. Although use of the Z transform allows greater numerical accuracy due to the spreading out of the passband poles, the prime reason for its use in this program is the mathematical expressions for elliptic filters are simpler in the Z domain than in the s domain.

Given a filter with equiripple passband extending from ω_A to ω_B, having NZ attenuation poles at the origin, N finite loss poles, and NINF attenuation poles at infinite frequency, the loss function in terms of Z is:

$$L(Z) = \left\{\frac{Z + \omega_B/\omega_A}{Z - \omega_B/\omega_A}\right\}^{\frac{NZ}{2}} \left\{\frac{Z+1}{Z-1}\right\}^{\frac{NINF}{2}} \prod_{i=1}^{N} \frac{Z + Z_i}{Z - Z_i} \qquad (2\text{-}13.2)$$

If L(Z) represents a normalized lowpass filter, then $\omega_A = 0$, $\omega_B = 1$, and NZ = 0. Letting $s = j\Omega$, Z and L(Z) become:

$$Z = (1 - 1/\Omega^2)^{\frac{1}{2}} \qquad (2\text{-}13.3)$$

$$L(Z) = \left\{\frac{Z+1}{Z-1}\right\}^{\frac{NINF}{2}} \prod_{i=1}^{N} \frac{Z + Z_i}{Z - Z_i} \qquad (2\text{-}13.4)$$

The attenuation function, $A(\Omega)$, is defined in terms of the loss function, L(Z), as follows:

$$A(\Omega) = 10 \cdot \log\left\{1 + \frac{\varepsilon^2}{4}\left(L(Z) + \frac{(-1)^{NINF}}{L(Z)}\right)^2\right\} \qquad (2\text{-}13.5)$$

$$\varepsilon^2 = 10^{0.1Amax} - 1 \qquad (2\text{-}13.6)$$

In the stopband, the attenuation function may be simplified:

$$A(\Omega) = 10 \log\left[1 + \frac{\varepsilon^2}{4}\left\{|L(Z)| + 1/\,|L(Z)|\right\}^2\right] \qquad (2\text{-}13.7)$$

The filter passband ripple (Amax) may sometimes be expressed in terms of a reflection coefficient, ρ. The relationship between these quantities is:

$$Amax = -10 \log (1 - \rho^2) \qquad (2\text{-}13.8)$$

Within the normalized lowpass passband ($\Omega<1$), Z becomes purely imaginary. Equation (2-13.4) may be rewritten in exponential form to eliminate the need for complex arithmetic:

$$L(Z) = e^{jB} \qquad (2\text{-}13.9)$$

where

$$B = \frac{NINF}{2} \tan^{-1}\left\{\frac{-2|Z|}{|Z|^2 - 1}\right\} + \sum_{i=1}^{N} \tan^{-1}\left\{\frac{-2|Z|Z_i}{|Z|^2 - Z_i^2}\right\} \qquad (2\text{-}13.10)$$

substituting Eq. (2-13.9) into (2-13.5) yields:

$$A(\Omega) = 10 \log (1 + \varepsilon^2 \cos^2 B) \text{ for NINF even,} \qquad (2\text{-}13.11)$$

and

$$A(\Omega) = 10 \log (1 + \varepsilon^2 \sin^2 B) \text{ for NINF odd.} \qquad (2\text{-}13.12)$$

The program uses Eqs. (2-13.3) through (2-13.12) to find the filter loss at any frequency. Two ancillary relations are used to convert unnormalized bandpass or bandstop frequencies to the normalized lowpass frequency, Ω. Lowpass and highpass filters are only special cases of bandpass and bandstop filters respectively, in that the center frequency is zero. These two ancillary equations are:

Bandpass to normalized lowpass

$$\Omega_{BP} = \frac{1}{BW}\left\{f - \frac{f_o^2}{f}\right\} \qquad (2\text{-}13.13)$$

where

BW = bandwidth

f_o = center frequency

Bandstop to normalized lowpass

$$\Omega_{BS} = 1/\Omega_{BP} \qquad (2\text{-}13.14)$$

Equation (2-13.4) will predict the stopband attenuation for even ordered elliptic filters of Cauer types A and B (the Möbius transformation - see previous program for description). The type A, even-ordered filter has no attenuation poles at infinite frequency, and can only be realized with mutual inductive coupling between filter sections, while the Möbius transformed pole locations (type B) can be realized with a ladder structure containing only L's and C's. The even ordered type B ladder structure possesses a double pole of attenuation at infinite frequency.

Equation (2-13.4) will not work with the pole locations resulting from a transformation to Cauer type C filters (equal resistive termination for even-ordered elliptic filters), i.e., one must use types A and B only. See Saal and Ulbrich [45] for details.

User Instructions

CHEBYSHEV PASSBAND - ARBITRARY STOPBAND RESPONSE				
load p_i	load NINF	load $+\rho$ or $-$Amax		
bandpass: f_o ↑ BW	bandstop: f_o ↑ BW	load f_{st}↑f_{sp}↑Δf	start sweep	calculate $f \rightarrow A(f)$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	If this program is being used concurrently with Program 2-11, the loss poles, Amax, and NINF are already stored by that program. Go to step 6 and continue.			
3	Load normalized loss pole frequencies	setup	f A	
		p_1	R/S	
		p_2	R/S	
		⋮	⋮	
		p_n	R/S	
4	Load number of loss poles at infinite frequency	NINF	f B	
5	Load either the reflection coefficient or the passband ripple in dB (related quantities). The program differentiates the quantities by sign. Both quantities are normally positive			
	Reflection coefficient	ρ	f C	Amax
	or			
	Passband ripple in dB (note sign)	$-$Amax	f C	Amax
6	Select filter type:			
	Bandpass or Lowpass:	f_o	ENT↑	
	The lowpass filter is a special case of the bandpass filter in that the center frequency is zero. The bandwidth is the lowpass cutoff frequency.	BW	A	
	Bandstop or Highpass:	f_o	ENT↑	
	The highpass filter is a special case of the bandstop filter in that the center frequency is zero. The bandwidth is the highpass cutoff frequency.	BW	B	

User Instructions

CHEBYSHEV PASSBAND - ARBITRARY STOPBAND RESPONSE		
	CONTINUED	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
7	Load denormalized stopband frequency and calculate stopband attenuation	f	E	A(f)
8	If a sweep of frequencies is desired:			
	a) Load sweep parameters	f_{start}	ENT↑	
	The frequency increment is either an	f_{stop}	ENT↑	
	additive delta, or a multiplicative delta depending upon lin/log sweep. If linear sweep is desired, then the increment should be entered as a negative quantity, i.e., for 100 Hz linear steps, the increment should be entered as -100. Whether the increment is linear or logrithmic, the sweep will be always in the direction of increasing frequency.	f_{incr}	C	
	b) Start sweep		D	f
				A(f)
				space
				f
				A(f)
				⋮
9	Go back to any step desired, modify, and rerun program			

Example 2-13.1

An elliptic bandpass filter is required to pass frequencies between 5 kHz and 15 kHz with 0.0436 dB ripple or less (10% reflection coefficient), and reject frequencies lying outside a 4.1 kHz to 19 kHz band by at least 60 dB. Find the minimum filter order that will satisfy these requirements, and predict the stopband response.

The center frequency is the geometric mean of the upper and lower passband edge frequencies. Likewise, the stopband edge frequencies must be geometrically symmetrical about the center frequency. In this example, the above frequencies do not satisfy this requirement, hence, the narrowest stopband with geometric symmetry must be defined. The filter center frequency is calculated from the passband edge frequencies:

$$f_o = (5000 \cdot 15000)^{\frac{1}{2}} = 8660.25 \text{ Hz}$$

The narrowest stopband may be found by calculating the geometrical mating frequencies to the given stopband frequencies, and taking the narrowest set:

$$f_u = f_o^{\,2}/4100 = 18292.68 \text{ Hz}$$

$$f_L = f_o^{\,2}/19000 = 3947.37 \text{ Hz}$$

The narrowest stopband is 4100 Hz to 18292.68 Hz for a stopband width of 14192.68 Hz.

The stopband and passband data are loaded into Program 2-12 to find the minimum filter order and loss pole locations. Because bandpass data was loaded, the loss pole frequencies that are output represent loss pole bandwidths, or the separation of loss pole frequencies in the upper and lower stopbands that are geometrically related to the filter center frequency. To convert these bandwidths into loss pole frequencies, the subprogram contained in Program 2-1 can be used. These equivalent bandpass loss pole frequencies are not necessary for proper operation of this program, but are calculated for information only. They can also be useful when tuning the final filter. All normalized loss pole information is automatically stored by Program 2-12 for use by this program.

Example 2-13.1 continued

```
Load Program 2-12 and calculate filter order and loss poles.
          .0436 GSBA  load Amax
          60.00 GSBa  load Amin
       10000.00 GSBE  load fmax (passband bandwidth)
       14192.68 GSBb  load fmin (stopband minimum bandwidth)

                GSBC  calculate minimum filter order

           6.72  ***
           7.00  ***  minimum integral filter order, n

                GSBD  calculate loss pole bandwidths

    28.69564+03  ***  loss pole bandwidth #1
    17.08915+03  ***  loss pole bandwidth #2
    14.44925+03  ***  loss pole bandwidth #3
--------------------------------------------------------------
Load Program 2-1 to calculate loss pole locations from
loss pole bandwidths.
       28695.64 ENT↑  load loss pole bandwidth #1
        8660.25 GSBd  load fo
       31106.65  ***  upper BP loss pole frequency #1
        2411.05  ***  lower BP loss pole frequency #1

       17089.19 ENT↑  load loss pole bandwidth #2
        8660.25 GSBd  load fo
       20710.53  ***  upper BP loss pole frequency #2
        3621.34  ***  lower BP loss pole frequency #2

       14449.25 ENT↑  load loss pole bandwidth #3
        8660.25 GSBd  load fo
       18502.71  ***  upper BP loss pole frequency #3
        4053.46  ***  lower BP loss pole frequency #3
--------------------------------------------------------------
Load this program (Program 2-13) and calculate filter response.

        8660.25 ENT↑  load fo
       10000.00 GSBA  load passband width and select bandpass

        2000.00 ENT↑  load f-start
        5000.00 ENT↑  load f-stop
        -200.00 GSBC  load f-increment (a negative value
                          means linear sweep increments)
                GSBD  start sweep:
                          the output is on the next page.
(sweep step size changes were made between output segments)
```

PROGRAM OUTPUT FOR EXAMPLE 2-13.1

LOWER STOPBAND f	A(f) dB	PASSBAND f	A(f)	PASSBAND f	A(f)	UPPER STOPBAND f	A(f)	UPPER STOPBAND f	A(f)
2000.00	68.02	5000.0000	0.0436	10000.0000	0.0434	15000.00	0.04	30000.00	79.85
2200.00	73.03	5500.0000	0.0020	10500.0000	0.0342	16000.00	12.98	31000.00	100.57
2400.00	98.12	6000.0000	0.0389	11000.0000	0.0115	17000.00	31.72	32000.00	82.55
2600.00	73.25	6500.0000	0.0002	11500.0000	0.0000	18000.00	52.97	33000.00	76.47
2800.00	67.19	7000.0000	0.0248	12000.0000	0.0144	19000.00	64.35	34000.00	73.25
3000.00	64.45	7500.0000	0.0434	12500.0000	0.0389	20000.00	68.65	35000.00	71.15
3200.00	63.90	8000.0000	0.0234	13000.0000	0.0385	21000.00	77.55	36000.00	69.63
3400.00	66.45	8500.0000	0.0016	13500.0000	0.0082	22000.00	66.70	37000.00	68.49
3600.00	84.71	9000.0000	0.0065	14000.0000	0.0090	23000.00	64.24	38000.00	67.60
3800.00	66.03	9500.0000	0.0293	14500.0000	0.0430	24000.00	63.83	39000.00	66.89
4000.00	67.53			15000.0000	0.0436	25000.00	64.45	40000.00	66.31
4200.00	49.19					26000.00	65.75	41000.00	65.84
4400.00	32.55					27000.00	67.65	42000.00	65.45
4600.00	18.89					28000.00	70.25	43000.00	65.13
4800.00	5.63					29000.00	73.91	44000.00	64.66
5000.00	0.04							45000.00	64.64

NOTE:
The display was changed manually to DSP4 for the passband printout, then changed back to DSP2 for the upper stopband output.

Example 2-13.2

Compute the minimum stopband attenuation of an eleventh order, 20% reflection coefficient, 75 degree modular angle elliptic filter (see p. 326 of Saal and Ulbrich [45]).

Load Program 2-12 and calculate filter order and loss pole locations.

```
     1.00 ENT↑  }
      .20 X²    }
          -     }  calculate Amax = 10 log (1 - ρ²)
          LOG   }
    10.00 ×     }
          CHS   }
0.1772877 ***   }

          GSBH  load Amax
   100.00 GSBa  load dummy Amin large enough to cause "error" halt
     1.00 GSBB  load normalized passband edge
    75.00 D→R   }
          SIN   }  calculate stopband edge from modular angle
          1/X   }
 1.035276 ***   }
          GSBb  load normalized stopband edge

          GSBC  start filter order calculation (computes K(k))
          ERROR program halt since L⁻¹ is too small

      11. GSBc  load desired filter order

          GSBD  output loss pole locations and store for next program
2.222241138 ***
1.[illegible]44326902 ***
1.126526299 ***
1.05[illegible]341 ***
1.0375135[illegible] ***
```

Load this program (Program 2-13) and calculate filter response.

```
 0.000 ENT↑  load filter center frequency (lowpass)
 1.000 GSBH  load bandwidth (normalized for this example)

 1.000 GSBb  load number of poles at infinity

75.000 SIN  }  calculate normalized stopband edge frequency
       1/X  }
       GSBE  calculate stopband loss at this frequency
60.606 ***   minimum stopband loss in dB (Amin defined at fmin)
```

Program Listing I

Step	Key	Comment
001	*LBLa	SETUP LOSS POLE ENTRY
002	1	
003	3	initialize index register
004	STOI	
005	SF2	indicate initialization reqd
006	*LBL0	
007	R/S	enter normalized loss pole
008	ISZI	
009	STOi	
010	GTO0	
011	*LBLb	LOAD NINF, the number of
012	STOD	lowpass loss poles at
013	GTO6	infinite frequency
014	*LBLc	LOAD ρ or -Amax
015	X<0?	if negative entry, jump
016	GTO1	
017	X²	calculate:
018	CHS	Amax $= \lvert 10\log(1-\rho^2)\rvert$
019	GSB7	
020	*LBL1	
021	ABS	store $\lvert$Amax$\rvert$
022	STO0	
023	GTO6	goto space and return
024	*LBLA	LOAD f_0 & BW for bandpass
025	CF0	indicate bandpass
026	GTO1	
027	*LBLB	LOAD f_0 & BW for bandstop
028	SF0	indicate bandstop
029	*LBL1	store BW (bandwidth)
030	STO9	
031	R↓	
032	X²	form and store f_0^2
033	STO8	
034	GTO6	goto space and return
035	*LBLC	LOAD f-start, f-stop, Δf
036	P⇄S	store f-increment (Δf)
037	STO1	
038	R↓	store f-stop
039	STO2	
040	R↓	store f-start
041	STO0	
042	P⇄S	restore register order
043	GTO6	goto space and return
044	*LBLD	START SWEEP
045	P⇄S	
046	RCL0	recall and print
047	P⇄S	present frequency
048	PRTX	
049	GSBE	calculate and print A(f)
050	P⇄S	recall frequency increment
051	RCL1	
052	X<0?	
053	ST-0	if increment negative,
054	X<0?	use additive delta
055	GTO1	
056	STx0	if plus, use product delta
057	*LBL1	
058	RCL2	
059	RCL0	test for loop exit
060	P⇄S	
061	X≤Y?	
062	GTOD	
063	RTN	
064	*LBLE	LOAD f, calculate A(f)
065	STO6	temporarily store f
066	F2?	if first time through here,
067	GSB8	goto initialization routine
068	RCL6	
069	RCL8	calculate:
070	RCL6	
071	÷	$\Omega = \frac{1}{BW}\left[f - \frac{f_0^2}{f}\right]$
072	-	
073	ABS	
074	RCL9	
075	÷	
076	X=0?	Ω = 0 escape
077	RTN	
078	F0?	if bandstop, form inverse
079	1/X	
080	EEX	test for passband (Ω < 1)
081	CF3	
082	X>Y?	set flag 3 if passband
083	SF3	
084	X⇄Y	
085	X²	
086	1/X	form and store:
087	-	
088	ABS	$\lvert Z\rvert = (\lvert 1 - \frac{1}{\Omega}\rvert)^{\frac{1}{2}}$
089	√X	
090	STO6	
091	F3?	jump if in passband
092	GTO3	
093	EEX	stopband attenuation,
094	+	form and store:
095	RCL6	
096	EEX	$\left[\frac{Z+1}{Z-1}\right]^{NINF/2}$
097	-	
098	÷	
099	ABS	
100	RCLD	
101	Yˣ	beginning of L(Z) calc
102	√X	
103	STO5	
104	RCL7	
105	STOI	initialize index register
106	*LBL2	L(Z) calculation loop
107	ISZI	
108	RCL6	calculate $\frac{Z+Z_i}{Z-Z_i}$
109	RCLi	
110	+	
111	RCL6	
112	RCLi	

REGISTERS

0 Amax	1 $\epsilon^2/4$	2	3	4	5 L	6 scratch	7 "13"	8 f_0^2	9 BW
S0 present freq	S1 freq incr	S2 stop freq	S3	S4 ←	S5 loss pole	S6 storage	S7 registers	S8	S9 →

A ← loss pole	B storage →	C 10^{-9}	D NINF	E I_{max}	I storage register index

Program Listing II

NOTE FLAG SET STATUS

Step	Key	Comment
113	-	
114	÷	
115	STx5	form running product of L(Z)
116	RCLI	
117	RCLE	test for loop exit
118	X≠Y?	
119	GTO2	
120	RCL5	
121	1/X	form (L + 1/L)
122	LSTX	
123	GTO5	goto output routine
124	*LBL3	passband attenuation calc
125	ENT↑	
126	+	
127	CHS	
128	RCL6	form and store:
129	X²	
130	EEX	$\frac{NINF}{2}\beta_{INF}$;
131	-	
132	→P	$\beta_{INF} = \tan^{-1}\left[\frac{-2\lvert Z\rvert}{\lvert Z\rvert^2 - 1}\right]$
133	X⇄Y	
134	2	
135	÷	
136	RCLD	
137	x	
138	STO5	
139	RCL7	
140	STOI	initialize index register
141	SF1	
142	RCLD	
143	2	set flag 1 if NINF is odd
144	÷	
145	FRC	
146	X=0?	
147	CF1	
148	*LBL4	L(Z) calculation loop
149	ISZI	increment index register
150	RCL6	
151	RCLi	
152	x	form β_i:
153	ENT↑	$\beta_i = \tan^{-1}\frac{-2\lvert Z\rvert Z_i}{\lvert Z\rvert^2 - Z_i^2}$
154	+	
155	CHS	
156	RCL6	
157	X²	
158	RCLi	
159	X²	
160	-	
161	→P	
162	X⇄Y	
163	ST+5	add β_i to running sum
164	RCLI	
165	RCLE	test for loop exit
166	X≠Y?	
167	GTO4	
168	RCL5	recall β
169	EEX	
170	→R	form sin β, and cos β
171	F1?	recall sin β→R_x if NINF odd
172	X⇄Y	
173	ENT↑	prepare to double Rx
174	*LBL5	Output routine; form
175	+	$\epsilon^2/4\,(L+1/L)^2$ if stopband
176	X²	
177	RCL1	$\epsilon^2 \frac{\sin^2\beta}{\cos^2\beta}$ if passband
178	x	
179	GSB7	calculate and print
180	RND	$10\log(1 + R_x)$
181	PRTX	
182	*LBL6	space and return subroutine
183	SPC	
184	RTN	
185	*LBL7	subroutine to calculate:
186	EEX	
187	+	$10\log(1 + (\cdot))$
188	LOG	
189	EEX	
190	1	
191	x	
192	RTN	
193	*LBL8	initialization routine
194	RCLI	store highest loss pole
195	STOE	register number
196	RCL0	
197	EEX	calculate and store:
198	1	
199	÷	
200	10ˣ	$\frac{\epsilon^2}{4} = \frac{10^{0.1Amax} - 1}{4}$
201	EEX	
202	-	
203	4	
204	÷	
205	STO1	
206	1	store index register
207	3	initialization, and
208	STO7	initialize index register
209	STOI	
210	*LBL9	loss pole Z transform loop
211	ISZI	increment index register
212	EEX	calculate and store:
213	RCLi	
214	X²	$Z_i = (1 - 1/(p_i)^2)^{\frac{1}{2}}$
215	1/X	where p_i are the normalized
216	-	loss pole frequencies
217	√X	
218	STOi	
219	RCLI	test for loop exit
220	RCLE	
221	X≠Y?	
222	GTO9	
223	RTN	return to main program

LABELS

A BANDPASS f_0 ∣ BW	B BANDSTOP f_0 ∣ BW	C f_{st} ∣ f_{sp} ∣ f	D START SWEEP	E f→A(f)
a losspole entry	b load NINF	c load ρ or -Amax	d	e
0 losspole ent loop	1 local lbl	2 L(Z) stop band	3 passband atten	4 L(Z) pass band
5 output routine	6 space & return	7 A(f)	8 init	9 losspole Z xfm

FLAGS

0	bandstop
1	NINF odd
2	init
3	passband

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	DEG ■	FIX ■
1		■	GRAD	SCI
2	■		RAD	ENG
3				n 2

PROGRAM 2-14 POLE AND ZERO LOCATIONS OF A FILTER WITH CHEBYSHEV PASSBAND AND ARBITRARY STOPBAND LOSS POLE LOCATIONS.

Program Description and Equations Used

This program calculates the complex zero locations of the filter transfer function, $H(s) = E_{in}/E_{out}$, from the loss pole frequencies (frequencies of infinite attenuation). The zero locations are also called the natural modes of H(s). The pole locations of H(s), are the loss pole frequencies and lie on the $j\omega$ axis. The transmission function, T(s), is the reciprocal of the filter transfer function, and may be more familiar to some readers. When active elliptic filters are being designed [35], one approach is to divide the transmission function into bi-quadratic factors with each factor (second order pole pair, and second order zero pair) being synthesized with a separate active network [38].

The loss pole frequencies can be supplied by the user in the case of arbitrary stopband, equiripple passband filters, or can be generated by Program 2-12 for elliptic filters (equiripple stopband and passband).

This program works in the Z-domain to spread out the pole and zero frequencies, and enhance the numerical accuracy of the final output. The s-domain frequencies are Z transformed using Eq. (2-14.1), which is the normalized lowpass form of the more generalized Z transform.

$$Z^2 = 1 + \frac{1}{s^2}\bigg|_{s=j\omega} = 1 - \frac{1}{\omega^2} \qquad (2\text{-}14.1)$$

The filter transfer function is a rational function, i.e., it is a ratio of polynomials:

$$H(s) = \frac{e(s)}{q(s)} \qquad (2\text{-}14.2)$$

This transfer function is related to the filter characteristic function, K(s), by the Feldtkeller equation:

$$H(s)H(-s) = 1 + K(s)K(-s) \qquad (2\text{-}14.3)$$

where the characteristic function has been defined in terms of the Chebyshev rational function, R(x,L), by Eq. (2-12.3), and also is a ratio of polynomials:

$$K(s) = \frac{f(s)}{q(s)} \qquad (2\text{-}14.4)$$

Expanding the Feldtkeller equation to remove the denominator polynomial, q(s), yields:

$$e(s)e(-s) = q(s)q(-s) + f(s)f(-s) \qquad (2\text{-}14.5)$$

If the normalized lowpass Z transformation of these s-domain polynomials are defined by:

$$F(Z) \Leftrightarrow f(s)/s^m \qquad (2\text{-}14.6)$$

$$Q(Z) \Leftrightarrow q(s)/s^m \qquad (2\text{-}14.7)$$

where

$$Z_i^2 = 1 + \left(\frac{\omega_i}{s}\right)^2$$

$$m = NINF + N \qquad (2\text{-}14.8)$$

$$NINF = \text{number of attenuation poles at } \infty \qquad (2\text{-}14.9)$$

$$N = \text{number of finite loss pole freqs} \qquad (2\text{-}14.10)$$

then the Z transform equivalent of Eq. (2-14.5) becomes:

$$E(Z)E^*(Z) = Q^2(Z) + F^2(Z) \qquad (2\text{-}14.11)$$

where

$$E(Z) \Leftrightarrow e(s)/s^m \qquad (2\text{-}14.11a)$$

$$E^*(Z) \Leftrightarrow e(-s)/s^m \qquad (2\text{-}14.11b)$$

The derivation of $Q^2(Z)$ and $F^2(Z)$ in terms of the Z transformed loss pole frequencies, Z_i, is done later and the results brought forward:

$$Q^2(Z) = (1-Z^2)^{NINF} \prod_{i=1}^{N} (Z^2 - Z_i^2)^2 \qquad (2\text{-}14.12)$$

$$F^2(Z) = \varepsilon^2 (Ev\ A(Z))^2 \qquad (2\text{-}14.13)$$

$$A(Z) = (Z + 1)^{NINF} \prod_{i=1}^{N} (Z + Z_i)^2 \qquad (2\text{-}14.14)$$

The program Z transforms the loss pole frequencies using Eq. (2-14.1) then forms E(Z)E*(Z) using Eqs. (2-14.11), (2-14.12), (2-14.13), and (2-14.14). The roots of E(Z)E*(Z) are found using the secant iteration method (described later), and exist as quads, i.e.:

$$(Z+\sigma+j\omega)(Z+\sigma-j\omega)(Z-\sigma+j\omega)(Z-\sigma-j\omega) = Z^4 + pZ^2 + q \qquad (2\text{-}14.15)$$

Equation (2-14.1) may be used in reverse to convert Eq. (2-14.15) to the s-domain equivalent. The right half s plane (RHP) poles are assigned to e(-s), and the LHP poles assigned to e(s). These LHP poles represent the natural modes of the filter, and may be defined by a natural frequency, ω_n, and a quality factor, Q:

$$\omega_n = (1 + p + q)^{1/2} \qquad (2\text{-}14.16)$$

$$Q = \left[2\left\{1 - (1 + \tfrac{p}{2})(1 + p + q)\right\}\right]^{1/2} \qquad (2\text{-}14.17)$$

The natural frequency and Q represent the program output.

User Instructions

CARD # 1, CARD # 2

TRANSFER FUNCTION ZEROS FROM LOSS POLE LOCATIONS

				START

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
	NOTE: This program takes loss pole frequencies stored in registers S4 through S8. Program 2-11 automatically stores the loss pole frequencies in these registers. If the loss pole frequencies are provided by the user, they should be loaded before proceeding.			
1	Load both sides of program card one			
2	Start program execution		E	flashing display
3	Insert second card into card reader, this card will be read by the program at the appropriate time. If the card is not inserted, the display will flash when the second card is to be read.			
4	Read both sides of second program card			
	The program execution will automatically			
	resume.			ω_{n_N}
	If the first program is halted (R/S key)			Q_N
	when the display flashes, the second program			space
	execution may be resumed by depressing key "E"			$\omega_{n_{N-1}}$
	after the second card loading.			Q_{N-1}
				⋮
				⋮
				ω_{n_1}
				Q_1
				space
				σ_0 (if odd)

Example 2-14.1

Find the natural modes for the elliptic filter given in Example 2-12.2.

Load Program 2-11 and calculate loss pole frequencies.

```
    1.00 ENT↑  }
     .20   X²  }
            -  }  convert 20% reflection coefficient into
          LOG  }  passband ripple in dB using:
   10.00    x  }      Ap_dB = -10log(1-ρ²)
          CHS  }
         DSP6  }
0.177288  ***     Ap_dB, passband ripple in dB
         DSP2
         GSBA
   78.00 GSBa     load stopband attenuation reqd, As_dB
    1.00 GSBB     load normalized cutoff frequency
    2.00 GSBb     load normalized minimum stopband frequency
         GSBC     calculate minimum filter order

    5.90  ***     calculated filter order
    6.00  ***     nearest integral filter order

         GSBD     calculate loss pole frequencies
7.235803+00  ***  }  untransformed even order
2.732051+00  ***  }  loss pole frequencies
2.061105+00  ***  }

2.922132+00  ***  }  Möbius transformed loss pole frequencies
2.129549+00  ***  }
```

Load this program (Program 2-14) and calculate the natural modes.

```
         GSBE     start E(Z)E*(Z) calculation

0.83278696  ***  ωn1  }
1.57099957  ***  Q1   }
                      }
1.03809259  ***  ωn2  }  complex zero locations describing
5.89989169  ***  Q2   }  natural modes
                      }
0.50079327  ***  ωn3  }
0.62665941  ***  Q3   }
```

The complex zero locations may be converted from the ω_n and Q description to real and imaginary parts to enable checking results against elliptic filter tables (see p. 248 of Zverev [58]). Equations (2-9.1), (2-9.2), and (2-9.3) are used for the conversion.

1.57099957	ENT↑	load Q_1 and calculate θ_1
	+	
	1/X	
	COS^{-1}	
71.44174416	***	θ_1 (degrees)
.83278696	→R	load ω_{n_1} and calculate real and imag parts
0.26505003	***	σ_1
	X⇄Y	
0.78948249	***	ω_1
5.89989185	ENT↑	load Q_2 and calculate θ_2
	+	
	1/X	
	COS^{-1}	
85.13850531	***	θ_2 (degrees)
1.03805259	→R	load ω_{n_2} and calculate real and imag parts
0.08797556	***	σ_2
	X⇄Y	
1.03435603	***	ω_2
.62665941	ENT↑	load Q_3 and calculate θ_3
	÷	
	1/X	
	COS^{-1}	
37.07171237	***	θ_3 (degrees)
.50079327	→R	load ω_{n_3} and calculate real and imag parts
0.39957373	***	σ_3
	X⇄Y	
0.30188530	***	ω_3

Derivation of Equations Used

The characteristic function, K(s), is a ratio of polynomials as indicated by Eqs. (2-14.4) and (2-12.3). The denominator of this function is already known in terms of the loss pole frequencies. In low-pass form, this polynomial is:

$$q(s) = \prod_{i=1}^{N} (s^2 + \omega_i^2) \tag{2-14.18}$$

H(s), the filter transfer function, is described in terms of the polynomials of the characteristic function by Eqs. (2-14.2) and (2-14.5). Since H(s) describes a realizable transfer function, the zeros of H(s) must lie in the LHP. With this condition in mind, the LHP zeros of e(s) e(-s) are assigned to e(s) and the RHP zeros assigned to e(-s). This root splitting brings us to the concept of a quad. Assume that e(s) is represented by complex conjugate root pairs and a real root if e(s) is odd, i.e.,

$$e(s) = (s + \sigma_o) \prod_{i=1}^{N} \left\{ s^2 + s(2\sigma_i) + \sigma_i^2 + \omega_i^2 \right\} \tag{2-14.19}$$

Then the right half s-plane roots are represented by e(-s):

$$e(-s) = (-s + \sigma_o) \prod_{i=1}^{N} \left\{ s^2 - s(2\sigma_i) + \sigma_i^2 + \omega_i^2 \right\} \tag{2-14.20}$$

hence:

$$e(s)\, e(-s) = (-s^2 + \sigma_o^2) \prod_{i=1}^{N} \left\{ s^4 + s^2\, 2(\omega_i^2 - \sigma_i^2) + (\omega_i^2 + \sigma_i^2)^2 \right\} \tag{2-14.21}$$

This concept is illustrated in Fig. 2-14.1.

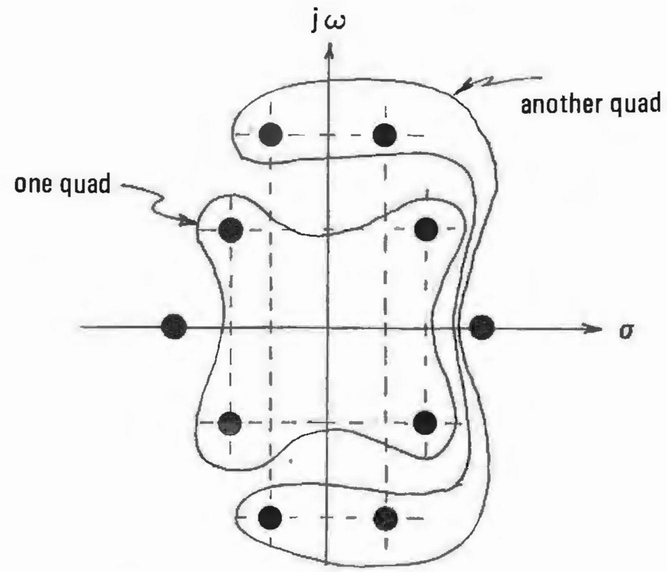


Figure 2-14.1 Concept of a quad.

The importance of this concept is once one root of e(s) e(-s) is found, three other roots of the quad are also defined, and may be removed to reduce the order of e(s) e(-s) by four.

The Characteristic Function in Terms of the Transformed Variable

The actual finding of the polynomials of H(s) is done in the Z-plane rather than the s-plane for two reasons: 1) The solution is numerically more accurate because the roots are spread out and the small difference between big numbers problem is much reduced. 2) The expressions for $F^2(Z)$ and $Q^2(Z)$ are much simpler in terms of the transformed loss pole frequencies than are $f^2(s)$ and $q^2(s)$ in terms of the actual loss pole frequencies, ω_i. These transformations are defined as follows:

$$F(Z) \Leftrightarrow \frac{f(s)}{(s^2+\omega_a^2)^{m/2}} \qquad (2\text{-}14.22)$$

$$Q(Z) \Leftrightarrow \frac{q(s)}{(s^2+\omega_a^2)^{m/2}} \qquad (2\text{-}14.23)$$

where

$$Z^2 = \frac{s^2+\omega_b^2}{s^2+\omega_a^2} \qquad (2\text{-}14.24)$$

and

$$m = NZERO + NINF + N \qquad (2\text{-}14.25)$$

NZERO = number of attenuation poles at dc (equals zero for lowpass filters) (2-14.26)

NINF = number of attenuation poles at infinity (2-14.9)

N = number of finite loss pole frequencies (2-14.10)

In the normalized lowpass case, the lower bandedge transformation frequency, ω_a is dc ($\omega_a = 0$), and the upper bandedge transformation frequency, ω_b, is unity. Under these conditions the Z transformation becomes:

$$F(Z) \Leftrightarrow \frac{f(s)}{s^m} \qquad (2\text{-}14.6)$$

$$Q(Z) \Leftrightarrow \frac{q(s)}{s^m} \qquad (2\text{-}14.7)$$

with

$$Z^2 = 1 + 1/s^2, \text{ or for } s = j\omega,\ Z^2 = 1 - 1/\omega^2$$

The lowpass form of q(s) is given by Eq. (2-14.18):

$$q(s) = \prod_{i=1}^{N} (s^2 + \omega_i^2)$$

The Z transformed equivalent is:

$$Q(Z) \Leftrightarrow \frac{1}{s^m} \prod_{i=1}^{N} (s^2 + \omega_i^2) \qquad (2\text{-}14.27)$$

$$= \frac{1}{s^{NINF}} \prod_{i=1}^{N} \left(\frac{s^2+\omega_i^2}{s^2}\right) \qquad (2\text{-}14.28)$$

The filter poles can be found from the zeros of the attenuation function, Eq. (2-13.5), i.e.,

$$1 + \frac{\varepsilon^2}{4}\left\{L(Z) + \frac{(-1)^{NINF}}{L(Z)}\right\}^2 = 0 \qquad (2\text{-}14.29)$$

where L(Z) is defined by Eq. (2-13.4):

$$L(Z) = \left\{\frac{Z + 1}{Z - 1}\right\}^{\frac{NINF}{2}} \cdot \prod_{i=1}^{N} \frac{Z + Z_i}{Z - Z_i} \qquad (2\text{-}13.4)$$

then Q(Z), as defined by Eq. (2-14.28), is the common denominator for Eq. (2-14.29). The quantity inside the brackets of Eq. (2-14.29) can be written in terms of Q(Z) and A(Z) (Eq. (2-14.14)) as follows:

$$L(Z) + \frac{(-1)^{NINF}}{L(Z)} = \frac{A(Z) + (-1)^{NINF} \cdot A(-Z)}{Q(Z)} \qquad (2\text{-}14.30)$$

Fortunately, the sign of $(-1)^{NINF}$ causes the numerator to be an even polynomial in Z as is required for the resulting polynomials of the Chebyshev rational function to be Hurwitz.

Thus, the equation whose zeros are to be found is:

$$1 + \frac{\varepsilon^2}{4}\left\{\frac{A(Z) + (-1)^{NINF} \cdot A(-Z)}{Q(Z)}\right\}^2 = 0 \qquad (2\text{-}14.31)$$

Because of the even numerator polynomial, Eq. (2-14.31) becomes:

$$1 + \frac{\varepsilon^2}{4}\left\{\frac{2 \cdot Ev\,(A(Z))}{Q(Z)}\right\}^2 = 0 \qquad (2\text{-}14.32)$$

Cancelling out constants and placing the entire expression over a common denominator yields:

$$Q^2(Z) + \varepsilon^2\left\{Ev\,A(Z)\right\}^2 = 0 \qquad (2\text{-}14.33)$$

Substituting F(Z) from Eq. (2-14.13) results the desired expression for the transfer function zeros:

$$Q^2(Z) + F^2(Z) = 0 \qquad (2\text{-}14.34)$$

Secant iteration method

The secant iterative method finds the values for the variable, x, where the function f(x) = 0 (zeros of x). It is similar to the

Newton-Raphson method except the derivative of the function is numerically approximated from the present and past values of f(x):

$$x_{i+1} = x_i - f(x_i)\left\{\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right\} \qquad (2\text{-}14.35)$$

where x_i is the present estimate for the variable.

The iteration is continued until the correction term magnitude becomes smaller than a given error radius. For this program, that error radius is chosen to be 10^{-9}.

Two values for x are needed to start the secant iteration, a past value and a present value. In this program, the past value is chosen as 0 and the present value as 1460°. As the iteration starts, the method may not converge, but may get sent far away from the desired solution. This can happen if the present and past estimates lie on opposite sides of a saddle (see Fig. 2-14.3). To help force convergence, the magnitude of the correction radius is limited to 0.1. When the iteration starts, the estimates have a random nature, but can't get far away from the origin. As a zero is approached, the method rapidly converges.

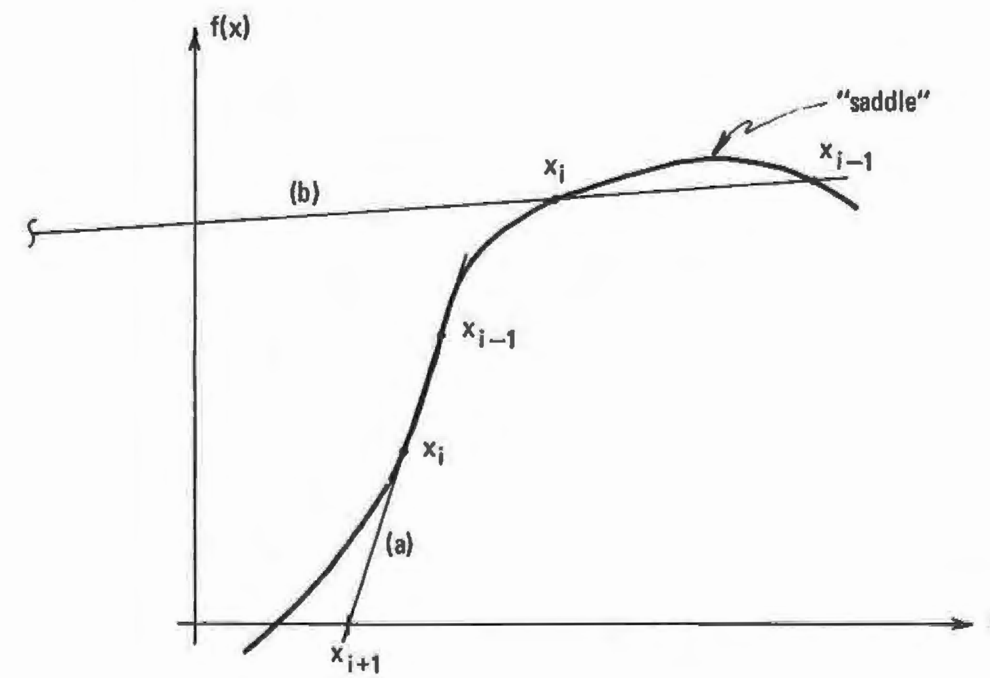


Figure 2-14.3 Secant method, two cases a) normal convergence, and b) divergence caused by the presence of a "saddle" in the function.

Figure 2-14.3 shows a two dimensional representation of the method, but in the present instance, the application is three dimensional because of the complex nature of the variable. As each complex zero is found, three others are defined automatically because of the quadrangle symmetry (quads) in the zeros of the filter transfer function (see Fig. 2-14.1), thus the order of the equation may be reduced by four through polynomial division. If the filter order is odd, a real zero exists in the transfer function. After all quads have been removed from the transfer function, the remainder will be the real zero. This technique is used herein, zeros are removed from E(Z)E*(Z) until a second order polynomial or less remains. If the filter order is even, no remainder exists, but if the filter order is odd, the second order remainder represents the LHP and RHP parts of the real zero. The RHP zero is discarded since it belongs to H(-s), and the LHP zero location is transformed back to the s-domain for output.

2-14, CARD 1

Program Listing I

ALGORITHM TO FORM E(Z)E*(Z) FROM $F^2(Z) + Q^2(Z)$

Key	Comment
*LBLE	START
RCLI	store index number of highest register w/ coef^s
STOB	
RCL0	calculate ϵ^2: $\epsilon^2 = 10^{0.1\epsilon dB} - 1$
EEX	
1	
÷	
10^x	
EEX	
-	
RCLD	store NINF + ϵ^2
+	
STOD	
1	initialize index register
3	
STOI	
*LBL0	Z transform loss pole freqs $Z_i^2 = 1 - 1/\omega_i^2$
ISZI	
EEX	
RCLi	
X^2	
1/X	
-	
STOi	
RCLI	test for loop exit
RCLB	
X≠Y?	
GTO0	
RCLD	set flag 2 if NINF = 2
INT	
2	
CF0	
X=Y?	
SF0	
EEX	start $Q^2(Z)$ calculation: form and store $(1 - Z^2)^{NINF}$ for NINF = 1 or 2
STOA	
CHS	
STOC	
F0?	
CHS	
STO1	
CHS	
STO0	
F0?	
GSB9	
1	initialize index register
3	
STOI	
*LBL1	$Q^2(Z)$ calculation loop $Q^2(Z) = (1-Z^2)^{NINF} \prod_{i=1}^{N} (Z^2 - Z_i^2)^2$, $-Z_i^2 \rightarrow R_C$
ISZI	
RCLi	
CHS	
STOC	
GSB9	
GSB9	
RCLI	
RCLB	test for loop exit
X≠Y?	
GTO1	
CLX	start A(Z) calculation
STOA	
3	initialize index register
STOI	
P⇄S	place $Q^2(Z)$ in secondary regs
EEX	initialize polynomial product registers
STO0	
RCLB	reduce highest loss pole register number by 10 to reflect P⇄S of registers
EEX	
1	
-	
STOB	
*LBL2	A(Z) calculation loop start $A(Z) = (Z+1)^{NINF} \prod_{i=1}^{N} (Z + Z_i)^2$, $Z_i \rightarrow R_C$
ISZI	
RCLi	
$\sqrt{X}$	
STOC	
GSB9	multiply existing polynomial product by $(Z + Z_i)^2$
GSB9	
RCLI	test for loop exit
RCLB	
X≠Y?	
GTO2	
EEX	setup to form $(Z + 1)^{NINF}$
STOC	
GSB9	multiply existing polynomial by $(Z + 1)^{NINF}$, (NINF = 1 or 2)
F0?	
GSB9	
EEX	calculate highest register number containing polynomial coefficients: 2B + 10 + F0 → RB
RCLB	
RCLB	
+	
5	
+	
F0?	
+	
STOB	
STOI	initialize index register
*LBL3	clear registers not containing polynomial coefficients
ISZI	
CLX	
STOi	
P⇄S	
STOi	
P⇄S	
RCLI	test for loop exit
1	
9	
X≠Y?	
GTO3	
RCL0	form $(Ev\ A(Z))^2$:
RCL2	
x	

REGISTERS

0	1	2	3	4	5	6	7	8	9
Q_0, A_0, A_0	Q_1, A_1, A_2	Q_2, A_2, A_4	Q_3, A_3, A_6	Q_4, A_4, A_8	Q_5, A_5, A_{10}	Q_6, A_6, A_{12}	Q_7, A_7, A_{14}	Q_8, A_8, A_{16}	Q_9, A_9, A_{18}

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Q_0, F_0	Q_1, F_1	Q_2, F_2	Q_3, F_3	Q_4, F_4, Z_1	Q_5, F_5, Z_2	Q_6, F_6, Z_3	Q_7, F_7, Z_4	Q_8, F_8	Q_9, F_9

A	B	C	D	E	I
Q & F index	highest reg. number used	Z_i^2	ϵ^2 + NINF	Z index	index

2–14, CARD 1

Program Listing II

Step	Key	Comments
113	STO1	$A_2' = 2A_0A_2$
114	ST+1	
115	RCL0	
116	RCL6	(NOTE: the primed coef's
117	x	represent the coefs of
118	RCL2	$A^2(Z)$. After this part
119	RCL4	of the program is done
120	x	the coefficients of
121	+	$A^2(Z)$ have replaced the
122	STO3	coeffients of A(Z).)
123	ST+3	$A_6' = 2(A_0A_6 + A_2A_4)$
124	RCL2	
125	RCL8	
126	x	
127	RCL4	$A_{10}' = 2(A_2A_8 + A_4A_6)$
128	RCL6	
129	x	
130	+	
131	STO5	
132	ST+5	
133	RCL6	
134	RCL8	
135	x	$A_{14}' = 2A_6A_8$
136	STO7	
137	ST+7	
138	6	initialize index register
139	STOI	
140	RCL8	$A_{16}' = A_8^2$
141	STx8	
142	RCL4	
143	x	$A_{12}' = A_6^2 + 2A_4A_8$
144	GSB7	
145	RCL2	
146	x	
147	RCL8	
148	√X	$A_8' = A_4^2 + 2(A_2A_6 + A_0A_8)$
149	RCL0	
150	x	
151	+	
152	GSB7	
153	RCL0	
154	x	$A_4' = A_2^2 + 2A_0A_4$
155	GSB7	
156	RCL0	$A_0' = A_0^2$
157	STx0	
158	RCLD	recall ϵ^2
159	FRC	
160	STx0	
161	STx1	form $F^2(Z) = \epsilon^2(Ev(A(Z)))^2$
162	STx2	
163	STx3	
164	STx4	
165	STx5	
166	STx6	
167	STx7	
168	STx8	

Step	Key	Comments
169	9	initialize index register
170	STOI	
171	*LBL4	
172	DSZI	form E(Z)E*(Z)
173	SF2	
174	RCLi	$E(Z)E^*(Z) = Q^2(Z) + F^2(Z)$
175	P⇄S	
176	ST+i	
177	P⇄S	
178	F2?	test for loop exit
179	GTO4	
180	*LBL5	wait loop for 2nd card read
181	PSE	
182	GTO5	
183	*LBL7	$A^2(Z)$ calculation subr
184	RCLi	forms:
185	STxi	
186	R↓	$R(i)^2 + 2(Rx)$, and returns
187	ST+i	
188	ST+i	
189	R↑	$R(i) \rightarrow Rx$
190	DSZI	
191	DSZI	
192	RTN	
193	RTN	
194	*LBL9	polynomial multiplication
195	SF2	flag 2 indicates 1st time
196	RCLA	initialize index register
197	X⇄I	with Q or F index
198	STOE	save existing index
199	*LBL8	polynomial mult loop
200	RCLi	
201	ISZI	
202	RCLC	$a_{k+1} = C \cdot a_{k+1} + a_k$
203	F2?	
204	CLX	$C = 0$ for $k = n$
205	STxi	
206	R↓	
207	ST+i	
208	CF1	decrement I register, and
209	DSZI	set flag 1 if I = 0
210	SF1	
211	DSZI	decrement I register
212	F1?	test for loop exit
213	GTO8	
214	RCLC	finish poly multiplication
215	STx0	$a_0 = C \cdot a_0$
216	RCLE	restore pre-existing index
217	STOI	
218	RCLA	increment F or Q index
219	EEX	
220	+	
221	STOA	
222	RTN	return to main program

LABELS					FLAGS	SET STATUS		
A	B	C	D	E START	0 NINF = 2	FLAGS	TRIG	DISP
a	b	c	d	e	1	ON OFF 0 ■ (OFF) 1 2 ■ (OFF) 3	DEG GRAD RAD	FIX ■ n 2
0 Z_i calc	1 $Q^2(Z)$	2 A(Z)	3 clear unused reg	4 E(Z)E*(Z)	2 I = ∅			
5 wait loop	6	7 $F^2(Z)$ subr	8 poly multiply	9 poly multiply	3			

2–14, CARD 2

Program Listing I

SECANT ITERATION TO FIND ROOTS OF E(Z)E*(Z)

Step	Key	Comments
001	*LBLE	START SECANT ITERATION
002	EEX	
003	CHS	set correction radius for
004	9	loop exit
005	STOE	
006	*LBLe	secant outer loop start
007	CLX	
008	STO0	set $Z_o = 0 + j0$
009	STO1	
010	STO5	
011	P⇄S	
012	RCL0	set $F(Z_o) = E_0 + j0$
013	P⇄S	
014	STO4	
015	6	set $Z_1 = 1 \angle 60°$
016	0	
017	ENT↑	
018	EEX	
019	→R	
020	STO2	
021	X⇄Y	
022	STO3	
023	*LBL0	prepare for polynomial
024	RCL2	evaluation:
025	RCL3	
026	x	form $Z^2 = (\sigma - j\omega)^2$;
027	ENT↑	
028	+	
029	STOD	$Im(Z^2) = 2\sigma\omega \rightarrow R_D \rightarrow R_7$
030	STO7	
031	RCL2	
032	X²	
033	RCL3	$Re(Z^2) = \sigma^2 - \omega^2 \rightarrow R_0 \rightarrow R_6$
034	X²	
035	-	
036	STOC	
037	STO6	
038	RCLB	set index to highest register
039	STOI	number that has coefficients
040	RCLi	start polynomial evaluation
041	STx6	by forming $E_{2n} \cdot Z^2$
042	STx7	
043	*LBL1	polynomial eval loop start,
044	DSZI	decrement register index
045	RCLi	recall E_{2k}
046	ST+6	add to calculation real part
047	RCLI	
048	EEX	test for loop exit
049	1	
050	X=Y?	
051	GTO2	
052	RCL6	perform complex multiply
053	RCLC	by Z^2 on the ongoing
054	x	calculation
055	RCL7	

Step	Key	Comments
056	RCLD	continue complex multiply
057	x	$(a + jb)(Re(Z^2) + jIm(Z^2)) =$
058	-	
059	RCL6	$a \cdot Re(Z^2) - b \cdot Im(Z^2) +$
060	RCLD	$j(a \cdot Im(Z^2) + b \cdot Re(Z^2))$
061	x	
062	RCL7	
063	RCLC	
064	x	
065	+	
066	STO7	
067	R↓	
068	STO6	
069	GTO1	
070	*LBL2	form Z estimate correction:
071	RCL7	
072	RCL6	
073	→P	$\Delta Z_k = F(Z_k)\left[\frac{Z_k - Z_{k-1}}{F(Z_k) - F(Z_{k-1})}\right]$
074	STO8	
075	X⇄Y	
076	STO9	
077	RCL3	
078	RCL1	
079	-	
080	RCL2	
081	RCL0	
082	-	
083	→P	
084	STx8	
085	X⇄Y	
086	ST+9	
087	RCL7	
088	RCL5	
089	-	
090	RCL6	
091	RCL4	
092	-	
093	→P	
094	X=0?	escape if $F(Z_k) - F(Z_{k-1}) = 0$
095	GTO3	
096	ST÷8	finish ΔZ_k calculation
097	X⇄Y	
098	ST-9	
099	RCL7	shift register contents,
100	STO5	Z_k becomes Z_{k-1}, and
101	RCL6	$F(Z_k)$ becomes $F(Z_{k-1})$ for
102	STO4	the next iteration
103	RCL3	
104	STO1	
105	RCL2	
106	STO0	
107	RCL8	recall $\lvert\Delta Z_k\rvert$
108	.	limit $\lvert\Delta Z_k\rvert$ to 0.1 to help
109	1	ensure convergence
110	X>Y?	

REGISTERS

0	1	2	3	4	5	6	7	8	9
Re Z_{k-1}	Im Z_{k-1}	Re Z_k	Im Z_k	ReF(Z_{k-1})	ImF(Z_{k-1})	ReF(Z_k)	ImF(Z_k)	scratch	scratch
S0 E_0	S1 E_2	S2 E_4	S3 E_6	S4 E_8	S5 E_{10}	S6 E_{12}	S7 E_{14}	S8 E_{16}	S9 E_{18}

A	B	C	D	E	I
	highest register #	Re(Z_k^2)	Im(Z_k^2)	error radius for loop exit	index

Program Listing II

```
111  R↓
112  RCL9   apply Z correction:
113  X⇄Y
114  →R        Z_{k+1} = Z_k - ΔZ
115  ST-2
116  X⇄Y
117  ST-3
118  RCL8
119  RCLE   test for loop exit
120  X≤Y?
121  GTO0
122  *LBL3  Z factor output and
123  RCL3   polynomial deflation by
124  X²     degree 4, i.e., the quad is:
125  RCL2
126  X²        Z^4 + pZ^2 + q
127  -
128  ENT↑      p = 2((Im Z_1)^2 - (Re Z_1)^2)
129  +
130  STOC   p → RC
131  RCL3
132  X²
133  RCL2
134  X²        q = ((Im Z_1)^2 + (Re Z_1)^2)^2
135  +
136  X²
137  STOD   q → RD
138  +      calculate and output the
139  EEX    s-plane LHP complex-conjugate
140  +      pole pair natural frequency:
141  √X
142  1/X       ω_n = (1 + p + q)^(-1/4)
143  √X
144  PRTX
145  EEX    calculate and output pole
146  LSTX   pair "Q":
147  RCLC
148  2
149  ÷
150  EEX
151  +         Q^2 = (2(1-(1+p/2)(1+p+q)))^-1
152  x
153  -
154  ENT↑
155  +
156  1/X
157  √X
158  PRTX   print Q
159  SPC
160  RCLB   set index to highest register
161  STOI   number that has coefficients
162  *LBL5  polynomial deflation loop
163  RCLi
164  RCLC
165  x
166  ISZI   let the primed values
167  RCLi   represent the coeffic-
168  RCLD   ients of the deflated
169  x      polynomial:
170  +
171  DSZI      E'_k = E_k - pE'_{k+2} - qE'_{k+4}
172  DSZI
173  ST-i
174  RCLI
175  1
176  3      test for loop exit
177  X≤Y?
178  GTO5
179  *LBL6  move coefficients down two
180  RCLi   register numbers in storage
181  DSZI   so E_0 resides in register S_0
182  DSZI
183  STOi
184  ISZI
185  ISZI
186  ISZI
187  RCLB
188  RCLI   test for loop exit
189  X≤Y?
190  GTO6
191  DSZI
192  CLX
193  STOi   clear top two registers.
194  DSZI
195  STOi
196  DSZI
197  RCLI   store index number of highest
198  STOB   register containing coefs
199  1
200  2      test for outer loop exit
201  X≤Y?
202  GTOe
203  EEX
204  1      if filter order is even,
205  RCLI   exit loop here
206  X≤Y?
207  RTN
208  RCLi   calculate real pole location
209  DSZI   in s-plane for odd ordered
210  RCLi   filter
211  X⇄Y
212  ÷         |σ_0| = (E_2/E_0 - 1)^(-1/2)
213  ABS
214  EEX
215  -
216  √X
217  1/X
218  PRTX
219  SPC
220  RTN
```

LABELS					FLAGS
A	B	C	D	E START	0
a	b	c	d	e main loop start	1
0 poly eval	1 poly eval	2 ΔZ calc	3 factor output	4	2
5 poly deflate	6 coef reorder	7	8	9	3

SET STATUS		
FLAGS (ON OFF)	TRIG	DISP
0	DEG	FIX ■
1	GRAD	SCI
2	RAD	ENG
3		n 8

PROGRAM 2-15 DARLINGTON'S ELLIPTIC FILTER ALGORITHMS.

Program Description and Equations Used

This program calculates the normalized transmission function pole and zero locations, and minimum stopband rejection for odd order elliptic filters. The program is based on Professor Sidney Darlington's paper which describes simple elliptic filter algorithms using transformations on elliptic sines and their moduli [18], and his unpublished HP-65 program on the same subject.

The output data is normalized to the passband cutoff frequency (f_p), however, the algorithm is normalized to the geometric mean of the passband and stopband edge frequencies as shown by Fig. 2-15.1.

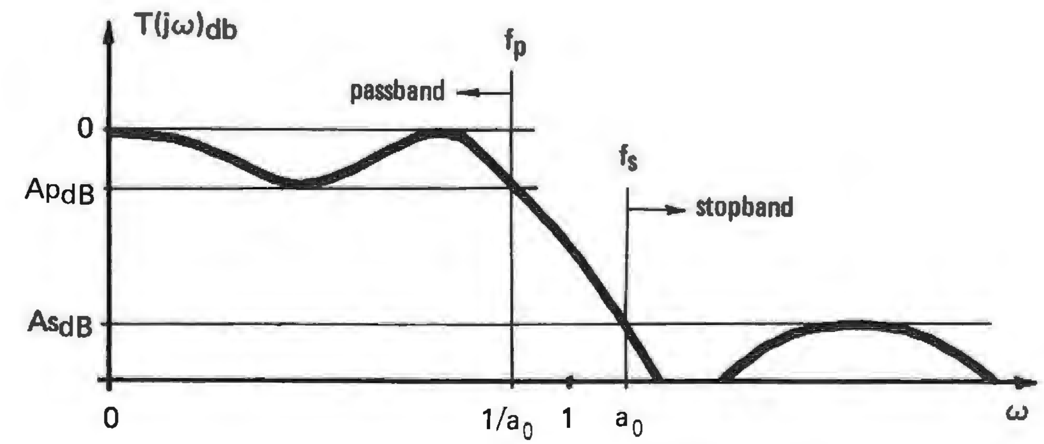


Figure 2-15.1 Definition of elliptic filter terms.

Thus, the transition ratio, λ , becomes:

$$\lambda = \frac{f_s}{f_p} = \frac{a_0}{1/a_0} = a_0^2 \qquad (2\text{-}15.1)$$

or

$$a_0 = \sqrt{\lambda} \qquad (2\text{-}15.2)$$

The filter transmission function, T(s), is the reciprocal of the filter transfer function, H(s), which is related to the filter characteristic function, K(s), through the Feldtkeller equation:

$$|T(j\omega)|^2 = \frac{\text{power out}}{\text{power in}} = \left|\frac{1}{H(j\omega)}\right|^2 = \frac{1}{1+\varepsilon^2|K(j\omega)|^2} \qquad (2\text{-}15.3)$$

where the characteristic function is the Chebyshev rational function described in Program 2-12. Darlington's algorithms are a very elegant way of approximating the Chebyshev rational function using simple recursive relationships. These relationships can also be used to find the LHP poles and zeros of:

$$T(s)T(-s) = \frac{1}{1+\varepsilon^2 K(s)K(-s)} \qquad (2\text{-}15.4)$$

Normalized transmission zero frequencies. If Yo represents geometrically normalized frequency (Fig. 2-15.1) and Y_{0k} $(k = 1, 2, \ldots, \frac{n-1}{2})$ represents the normalized transmission zero frequencies where n is the filter order, then the characteristic function for odd order, equiripple passband, lowpass filters is given by:

$$|K(Y_0)|^2 = |J_0 \cdot F_o(Y_0)|^2 \qquad (2\text{-}15.5)$$

where Jo is a constant and

$$F_o(Y_0) = Y_0 \prod_{k=1}^{(n-1)/2} \frac{1 - Y_{0k}^2 Y_0^2}{Y_0^2 - Y_{0k}^2} \qquad (2\text{-}15.6)$$

For the elliptic filter case (equal ripple passband and stopband):

$$\varepsilon^2 = 10^{0.1Ap_{dB}} - 1 \qquad (2\text{-}15.7)$$

$$J_0 = F_o(a_0) \qquad (2\text{-}15.8)$$

$$F_o\left(\frac{1}{Y_0}\right) = \frac{1}{F_o(Y_0)} \qquad (2\text{-}15.9)$$

These quantities and Y_{0k} may be found through recursive use of a variable transformation which spreads out the transition interval.

Let

$$a_{k+1} = a_k^2 + \sqrt{a_k^4 - 1} \qquad (2\text{-}15.10)$$

then, given a_0 as defined by Eq. (2-15.2), find and store a_1, a_2, a_3, and a_4. Four applications of the recursion formula will provide precision which will be calculator limited rather than algorithm limited (see p. 37 of [18]).

Let h represent the index for the transmission zero frequencies; h = 1, 2, ... , (n-1)/2, then let

$$Y_{4h} = \frac{a_4}{\cos\{(2h-1)(90/n)\}} \qquad (2\text{-}15.11)$$

and recursively calculate:

$$Y_{(k-1)h} = \frac{1}{2a_k}(Y_{kh} - 1/Y_{kh}) \qquad (2\text{-}15.12)$$

$k = 4, 3, 2, 1$

The transmission zero frequencies normalized with respect to the passband edge are:

$$a_o \cdot Y_{0h} \qquad (2\text{-}15.13)$$

Minimum stopband rejection. The minimum stopband rejection for elliptic filters first occurs at the stopband frequency edge (geometrically normalized frequency a_o) and may be found from Jo and Eqs. (2-15.4), (2-15.5), and (2-15.8), i.e.:

$$As_{dB} = 10 \log (1 + \varepsilon^2 J_0^2 J_0^2) \qquad (2\text{-}15.14)$$

Jo is found from another recursion relationship; let

$$J_4 \cong 2^{n-1} \cdot a_4^n = \frac{(2 \cdot a_4)^n}{2} \qquad (2\text{-}15.15)$$

then recursively calculate and store J_k's using:

$$J_{k-1} = \tfrac{1}{2}\sqrt{(J_k - 1/J_k)} \qquad (2\text{-}15.16)$$

$k = 4, 3, 2, 1$

Transmission function pole locations. Let s_{oh} represent the complex pole location, and let

$$J_o \cdot s_{oo} = 1/\varepsilon \qquad (2\text{-}15.17)$$

Then recursively calculate:

$$s_{(k+1)o} = J_k \cdot s_{ko} + \sqrt{(J_k \cdot s_{ko})^2 + 1} \qquad (2\text{-}15.18)$$

$$k = 0,\ 1,\ 2$$

As the index increases, the terms $J_k \cdot s_{ko}$ become numerically very large since the J_k's increase nearly geometrically for J_k large. To avoid numeric overflow (10^{99}) use:

$$s_{40} \cong 2 \cdot J_3 \cdot s_{30} \qquad (2\text{-}15.19)$$

Calculate and store:

$$s_{50} = \left\{ \frac{J_4}{s_{40}} + \sqrt{\left(\frac{J_4}{s_{40}}\right)^2 + 1} \right\}^{\frac{1}{n}} \qquad (2\text{-}15.20)$$

To calculate the pole locations, let:

$$s_{5h} = s_{50} \cdot e^{jh(\pi/n)} \qquad (2\text{-}15.21)$$

$$h = 0,\ 1,\ 2,\ \ldots,\ (n-1)/2$$

Using complex arithmetic, recursively calculate:

$$s_{(k-1)h} = \frac{1}{2 \cdot a_{k-1}} (s_{kh} - 1/s_{kh}) \qquad (2\text{-}15.22)$$

$$k = 5,\ 4,\ 3,\ 2,\ 1$$

The pole locations normalized to the passband edge are given by:

$$s_{oh} \cdot a_o \qquad (2\text{-}15.23)$$

The subroutine that calculates Eq. (2-15.22) may seem obscure to some readers. The particular coding that is used minimizes the amount of data that must undergo polar-to-rectangular and rectangular-to-polar conversions, and hence, maximizes the numerical accuracy of the routine. The normal format for the pole locations is polar as given by Eq. (2-15.21). In general, let:

$$s_{kh} = \rho_{kh} \cdot e^{j\beta_{kh}} \qquad (2\text{-}15.24)$$

In rectangular format, Eq. (2-15.24) becomes:

$$s_{kh} = \rho_{kh} \cos \beta_{kh} + j\, \rho_{kh} \sin \beta_{kh} \qquad (2\text{-}15.25)$$

For the reciprocal case, let:

$$\frac{1}{s_{kh}} = \frac{1}{\rho_{kh}}\, e^{-j\beta_{kh}} \qquad (2\text{-}15.26)$$

which using rectangular format becomes:

$$\frac{1}{s_{kh}} = \frac{1}{\rho_{kh}} \cos \beta_{kh} - j\, \frac{1}{\rho_{kh}} \sin \beta_{kh} \qquad (2\text{-}15.27)$$

hence,

$$s_{kh} - \frac{1}{s_{kh}} = \left(\rho_{kh} + \frac{1}{\rho_{kh}}\right) \cos \beta_{kh} + j\left(\rho_{kh} - \frac{1}{\rho_{kh}}\right) \sin \beta_{kh} \qquad (2\text{-}15.28)$$

or,

$$s_{kh} - \frac{1}{s_{kh}} = \left(1 + \frac{1}{\rho_{kh}^2}\right) \rho_{kh} \cos \beta_{kh} + j\left(1 - \frac{1}{\rho_{kh}^2}\right) \rho_{kh} \cdot \sin \beta_{kh} \qquad (2\text{-}15.29)$$

In Eq. (2-15.29), the terms $\rho_{kh} \cos \beta_{kh}$ and $\rho_{kh} \sin \beta_{kh}$ are the output components of a polar-to-rectangular conversion, and ρ_{kh} is saved in the last x register, and has not undergone any conversion. The stack is used to hold the intermediate parts of Eq. (2-15.29). A rectangular-to-polar conversion then completes the subroutine.

2-15 User Instructions

DARLINGTON'S ELLIPTIC FILTER ALGORITHMS				
load ±Ap_{dB}, or $-\rho$	load $\lambda = f_s/f_p$	load filter order (odd only)	calculate xmsn zero freqs & min loss	print ? / calculate pole locations

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Select print or R/S option (toggle)		f E	0 (R/S)
			f E	1 (print)
			f E	0 (R/S)
				⋮
3	Load passband ripple in dB or	Ap_{dB}	A	ϵ^2
	reflection coefficient	ρ	chs A	ϵ^2
4	Load stopband to passband frequency ratio	λ	B	
5	Load filter order (must be odd)	n	C	
6	Calculate normalized transmission zero			
	frequencies and minimum stopband loss		D	Ω_1
				Ω_2
				⋮
				$\Omega_{\frac{n-1}{2}}$
				As_{dB}
7	Calculate real and imaginary parts of			
	normalized transmission function poles		E	Re s_{o1}
				Im s_{o1}
				Re s_{o2}
				Im s_{o2}
				⋮
				Re s_{on}
				Im s_{on}

Example 2-15.1

Find the transmission function poles and zeros for a 9th order, elliptic filter having a 85° modular angle, and 50% reflection coefficient. Also calculate the minimum stopband attenuation in dB. Compare the results to the output of Program 2-11.

PROGRAM 2-15 INPUT	PROGRAM 2-15 OUTPUT
-.5 GSBA — load $-\rho$	GSBD — calc xmsn 0's
85. SIN — calculate	1.004553794+00 *** z_1
1/X — and load λ	1.014264420+00 *** z_2
1.003819836+00 ***	1.071140576+00 *** z_3
GSBE	1.449931830+00 *** z_4
9. GSBC — load n, the filter order	33.62429965+00 *** As_{dB} min
	GSBE — calc xmsn fcn poles
	372.8205714-03 *** Re p_o
	0.000000000+00 *** Im p_o
	182.7207935-03 *** Re p_1
	735.3062101-03 *** Im p_1
	41.73031646-03 *** Re p_2
	951.6234634-03 *** Im p_2
	7.669871966-03 *** Re p_3
	992.6118076-03 *** Im p_3
	1.191142461-03 *** Re p_4
	999.5098960-03 *** Im p_4

Load Program 2-12 and calculate transmission zeros (loss poles) for the same conditions.

PROGRAM 2–12 INPUT	PROGRAM 2–12 OUTPUT
1. ENT↑ .5 X² – calculate LOG and load 10. x Ap_{dB} for CHS 50% refl 1.249387366+00 *** coef GSBA 30. GSBa load As_{dB} 1. GSBB load f_p 85. SIN calculate and 1/X load f_s for 1.003815838+00 *** 85° modular GSBb angle	GSBC calculate filter order 8.352734615+00 *** calc order 9.000000000+00 *** integral order to meet specs GSBD calculate xmsn zero freq's 1.449931802+00 *** z_4 1.071140566+00 *** z_3 1.014284416+00 *** z_2 1.004553794+00 *** z_1

Comparing these results to those obtained from Program 2-15, differences exist in the 9th and 10th places sometimes. It is the author's opinion that the output from Program 2-12 is accurate to 2 parts in 10^{10}, since the elliptic sine algorithm and complete elliptic integral algorithm have been checked against the elliptic function tables in Abramowitz and Stegun [1] and disagree by at most one in the least significant digit of the HP-97 output (see Program 5-1 for details).

2-15 **Program Listing I**

Step	Key	Comment
001	*LBLA	LOAD Ap_{dB} or $-\rho$
002	X<0?	test for $-\rho$
003	GTOa	
004	EEX	calculate and store:
005	1	$\epsilon^2 = 10^{0.1 Ap_{dB}} - 1$
006	÷	
007	10ˣ	
008	EEX	
009	–	
010	STOA	
011	GTO9	
012	*LBLa	
013	X²	calculate and store:
014	CHS	
015	EEX	
016	+	
017	1/X	$\epsilon^2 = \frac{1}{1-\rho^2} - 1$
018	EEX	
019	–	
020	STOA	
021	GTO9	
022	*LBLB	LOAD λ, the stopband to
023	STOB	passband frequency ratio
024	GTO9	
025	*LBLC	LOAD n, the filter order
026	STOC	(must be odd)
027	GTO9	
028	*LBLD	CALC. xmsn zeros & As_{dB}
029	9	calculate and store:
030	0	$\frac{90}{n} \rightarrow R_E$
031	RCLC	
032	÷	
033	STOE	
034	EEX	
035	STOI	initialize k+1, 2h-1
036	STO9	
037	RCLB	$a_0 = \sqrt{f_s / f_p}$
038	√X	
039	GSB7	
040	GSB7	$a_{k+1} = a_k^2 + \sqrt{a_k^4 - 1}$
041	GSB7	k = 0, 1, 2, 3
042	GSB7	
043	*LBL0	
044	3	initialize k-1
045	STOI	
046	RCL4	calculate:
047	RCL9	
048	RCLE	$Y_{4h} = \frac{a_4}{\cos\{(2h-1)\frac{90}{n}\}}$
049	x	
050	COS	$h = 1, 2, \ldots, \frac{n-1}{2}$
051	÷	
052	SF0	indicate early subr exit
053	*LBL1	
054	GSB8	$Y_{(k-1)h} = \frac{1}{2a_k}\left(Y_{kh} - \frac{1}{Y_{kh}}\right)$
055	RCLi	k = 4, 3, 2
056	÷	
057	DSZI	test for loop exit
058	GTO1	
059	GSB8	Y_0 a_0 = xmsn zero freq
060	GSBd	print xmsn zero freq
061	2	increment 2h - 1
062	ST+9	
063	RCL9	
064	RCLC	
065	X>Y?	test for loop exit
066	GTO0	
067	GSB9	space if flag 1 is set
068	EEX	initialize k-1
069	STOI	
070	RCL4	calculate and store:
071	ENT↑	
072	+	$J_4 \cong \frac{(2a_4)^n}{2}$
073	RCLC	
074	Yˣ	
075	2	
076	÷	
077	P⇄S	
078	STO0	
079	CF0	indicate full subr
080	GSB8	$J_{k-1} = \sqrt{\frac{1}{2}\left(J_k - \frac{1}{J_k}\right)}$
081	GSB8	
082	GSB8	k = 4, 3, 2, 1
083	GSB8	
084	P⇄S	calculate and print:
085	X²	
086	X²	
087	RCLA	
088	x	
089	EEX	$As_{dB} = 10 \log(1 + \epsilon^2 J_0^4)$
090	+	
091	LOG	
092	1	
093	0	
094	x	
095	PRTX	
096	GTO9	goto space and return subr
097	*LBL7	subroutine to calculate:
098	X²	
099	ENT↑	
100	X²	
101	EEX	
102	–	
103	√X	$a_{k+1} = a_k^2 + \sqrt{a_k^4 - 1}$
104	+	
105	STOi	
106	ISZI	
107	RTN	

REGISTERS

0	1	2	3	4	5	6	7	8	9
a_0	a_1	a_2	a_3	a_4					index
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
J_4	J_3	J_2	J_1	J_0					

A	B	C	D	E	I
ϵ^2	λ	n	s	90/n	register index

Program Listing II

Step	Key	Comments
108	*LBL8	subroutine to calculate:
109	ENT↑	
110	1/X	$u_{k-1} \cdot c_{k-1} = \frac{1}{2}\left(u_k + \frac{1}{u_k}\right)$
111	+	
112	2	
113	÷	
114	F0?	test for early exit
115	RTN	
116	√X	
117	STOi	$J_{k-1} = \sqrt{\frac{1}{2}\left(J_k - \frac{1}{J_k}\right)}$
118	ISZI	
119	RTN	
120	*LBLE	CALCULATE POLE LOCATIONS
121	3	initialize 3-k
122	STOI	
123	RCLA	
124	√X	$J_0\, s_{00} = \frac{1}{\epsilon}$
125	1/X	
126	P⇄S	
127	*LBL2	calculate:
128	GSB6	$s_{(k+1)o} = J_k s_{ko} + \sqrt{J_k^2 s_{ko}^2 + 1}$
129	RCLi	k = 0, 1, 2,
130	×	
131	DSZI	test for loop exit
132	GTO2	
133	ENT↑	to avoid overflow, use:
134	+	$s_{4o} \cong 2\, J_3\, s_{3o}$
135	RCLi	calculate and store:
136	P⇄S	
137	X⇄Y	
138	÷	$s_{5o} = \left\{\frac{J_4}{s_{4o}} + \sqrt{\left(\frac{J_4}{s_{4o}}\right)^2 + 1}\right\}^{\frac{1}{n}}$
139	GSB6	
140	RCLC	
141	1/X	
142	Yˣ	
143	STOD	
144	CLX	initialize 2h
145	STO9	
146	*LBL4	pole location calc loop
147	4	initialize k-1
148	STOI	
149	RCL9	calc
150	RCLE	$s_{5h} = s_{5o}\, e^{jh\frac{\pi}{n}}$
151	×	$h = 0, 1, 2, \cdots, \frac{n-1}{2}$
152	RCLD	
153	*LBL3	$s_{(k-1)h} = \frac{1}{2 a_{k-1}}\left(s_{kh} - \frac{1}{s_{kh}}\right)$
154	GSB5	k = 5, 4, 3, 2
155	RCLi	
156	÷	
157	DSZI	test for loop exit
158	GTO3	
159	GSB5	finish pole location calculation and print pole locations
160	→R	
161	GSBa	
162	X⇄Y	
163	GSBa	
164	GSB[illegible]	
165	2	increment 2h
166	ST+9	
167	RCL9	test for loop exit
168	RCLC	
169	X>Y?	
170	GTO4	
171	*LBL[illegible]	space and return subr
172	F1?	space if flag 1 is set
173	SPC	
174	RTN	
175	[illegible]	R/S lookup
176	*LBL[illegible]	subroutine to calculate using complex arithmetic:
177	→P	
178	LSTX	
179	1/X	
180	[illegible]	
181	EEX	
182	-	
183	[illegible]	
184	LSTX	$s_{k-1} \cdot a_{k-1} = \frac{1}{2}\left(s_k - \frac{1}{s_k}\right)$
185	2	
186	+	
187	R↑	
188	×	
189	X⇄Y	
190	→R	
191	2	
192	÷	
193	RTN	
194	*LBL[illegible]	subroutine to calculate:
195	ENT↑	
196	X²	$s_{(k+1)o} = J_k s_{ko} + \sqrt{(J_k s_{ko})^2 + 1}$
197	EEX	
198	+	
199	√X	
200	+	
201	RTN	
202	*LBLa	print or R/S subroutine
203	F1?	
204	PRTX	print and return if flag 1 is set, otherwise
205	F1?	
206	RTN	
207	R/S	stop program execution
208	RTN	
209	*LBLd	PRINT-R/S TOGGLE
210	CF1	clear flag 1 and place a zero in the display
211	CLX	
212	RTN	
213	*LBLe	
214	SF1	set flag 1 and place a one in the display
215	EEX	
216	RTN	

LABELS					FLAGS
A LOAD A_{PdB} or $-\rho$	B LOAD λ	C LOAD n	D CALC zeros	E CALC poles	0 subr exit
a calc ϵ^2	b	c	d prt-R/S	e print?	1 print
0 h loop	1 k loop	2 k loop	3 k loop	4 h loop	2
5 subr	6 subr	7 subr	8 subr	9 subr	3

SET STATUS		
FLAGS	TRIG	DISP
ON OFF 0 1 2 3	DEG ■ GRAD RAD	FIX SCI ENG n____

Part 3

ELECTROMAGNETIC COMPONENT DESIGN

PROGRAM 3-1 FERROMAGNETIC CORE INDUCTOR DESIGN – MAGNETICS.

Program Description and Equations Used

This program calculates the various parameters relating to inductor or transformer design on closed magnetic cores. Given the core relative permeability (μ), the core length (ℓ_c), the core area (A), the air gap (ℓ_{air}), the required inductance (L), the dc current (I_{dc}), the applied ac voltage (E), and the excitation frequency (f), the program will calculate the number of turns required (N), the core H (oersteds) and B (gauss) resulting from the dc excitation, the ac excitation, and the total from both excitations. The dimensions of the core and air-gap can be entered in either centimeter or inch units. Program 3-2 will calculate the wire size and winding resistance given the window area and mean turn length. The program will also calculate the coil inductance if the number of turns, the core permeability and dimensions, and the air gap dimensions are given.

If the inductance in millihenries per 1000 turns is given (the A_L value) along with the core dimensions and permeability, the effective air gap will be calculated and stored in place of the given air gap. The inductance or turns, and core excitation will then be calculated on the basis of the calculated air gap.

The magnetic equations used are:

$$H = \frac{0.4\ \mu N I}{\ell_c + \mu_c \cdot \ell_{air}} \qquad (3.1.1)$$

$$E = 10^{-8} \cdot N \frac{d\phi}{dt} = 10^{-8} N A \frac{dB}{dt} \qquad (3\text{-}1.2)$$

where I is the current in the coil. Equation (3-1.2) can be rearranged to yield B, the core flux density:

$$B = \frac{10^8}{N A} \int E \cdot dt \qquad (3\text{-}1.3)$$

If $E = \sqrt{2} \cdot E_{rms} \cdot \sin(2\pi ft)$ is the sinewave excitation, then:

$$B_{peak} = \frac{10^8 \cdot E_{rms}}{\sqrt{2}\pi \ A N f} \tag{3-1.4}$$

If E is a symmetrical squarewave with voltage E_{pk} as shown by Fig. 3-1.1, then:

$$B_{peak} = \frac{10^8 \ Epk}{4 A N f} \tag{3-1.5}$$

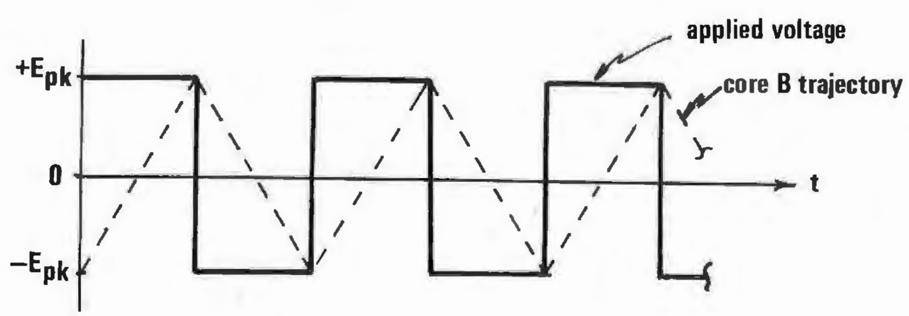


Figure 3-1.1 Square wave coil excitation and magnetic flux density trajectory.

Remembering the differential relationship between current and voltage in an inductor, $E = L(dI/dt)$, an expression can be derived relating the inductance, L, to the magnetic circuit quantities:

$$B = \mu H \tag{3-1.6}$$

From Eqs. (3-1.2) and (3-1.6):

$$E = 10^{-8} N \cdot A \mu \frac{dH}{dt} \tag{3-1.7}$$

From Eq. (3-1.1):

$$\frac{dH}{dt} = \frac{0.4\mu N}{\ell_c + \mu \ell_{air}} \cdot \frac{dI}{dt} \tag{3-1.8}$$

Combining Eqs. (3-1.7) and (3-1.8) yields the inductance expression:

$$E = \frac{0.4\pi N^2 A \cdot 10^{-8}}{\ell_c + \mu \ell_{air}} \cdot \frac{dI}{dt} \tag{3-1.9}$$

hence

$$L = \frac{0.4\pi N^2 \mu A \cdot 10^{-8}}{\ell_c + \mu \ell_{air}} \tag{3-1.10}$$

This equation may be rearranged to yield the equivalent air gap if the inductance per turn squared and core dimensions are known:

$$\ell_{air} = \frac{0.4\pi N^2 A \cdot 10^{-8}}{L} - \frac{\ell_c}{\mu}, \text{ cm} \tag{3-1.11}$$

Generally the inductance index in millihenries per 1000 turns is provided by the core manufacturer:

$$L^* = \text{millihenries per 1000 turns} \tag{3-1.12}$$

hence,

$$\ell_{air} = \frac{4\pi A}{L^*} - \frac{\ell_c}{\mu} \quad \text{cm} \tag{3-1.13}$$

Equation (3-1.10) can be rearranged to yield an expression for N, the number of turns, required to achieve a given inductance, L:

$$N = \left\{ L \frac{(\ell_c + \mu \ell_{air}) \cdot 10^8}{0.4 \pi \mu A} \right\}^{1/2} \tag{3-1.14}$$

The program uses these equations as follows: Labels "A," "a," "B," and "b" are used to load and store the core parameters. The actual stored parameters are in centimeters, and entries with inch units (Labels "A" and "B") are converted before storage. Label "C" uses Eq. (3-1.14) to calculate N given L. Label "c" uses Eq. (3-1.10) to calculate L given N. Label "d" uses Eq. (3-1.13) to calculate the equivalent air gap given the inductance index, L*. The new air gap dimension replaces the presently stored air gap dimension. Label "D" uses Eq. (3-1.1) to calculate the dc magnetizing force, H, given the dc current through the core. Since the number of turns are required for this calculation, the use of "C" or "c" must precede the use of "D." The dc flux density, B_{dc}, is calculated using Eq. (3-1.6). Label "E" uses Eq. (3-1.4) to calculate the peak core flux density given the ac coil excitation. The flux in the core will vary sinusoidally with sinusoidal excitation. The peak ac magnetizing force is calculated using Eq. (3-1.6). The peak ac and dc core magnetic parameters are added together and printed to provide the peak excitation in the core. The peak excitation should be kept below the magnetic saturation level of the core material for linear operation. Label "e" uses Eq. (3-1.5) to calculate peak core flux density from squarewave coil excitation, and provides a summary as above.

User Instructions

INDUCTOR DESIGN - MAGNETICS				
$\mu\uparrow \ell_c \uparrow A_c$ cm	ℓ_{air} cm	N → L	L*, $\frac{mh}{1000T}$	$E_{pk}\uparrow f_{Hz}$ H, B_{ac}, pk
$\mu\uparrow \ell_c \uparrow A_c$ inches	ℓ_{air} inches	L → N	I_{dc}→H, B_{dc}	$E_{rms}\uparrow f_{Hz}$ H, B_{ac}, pk

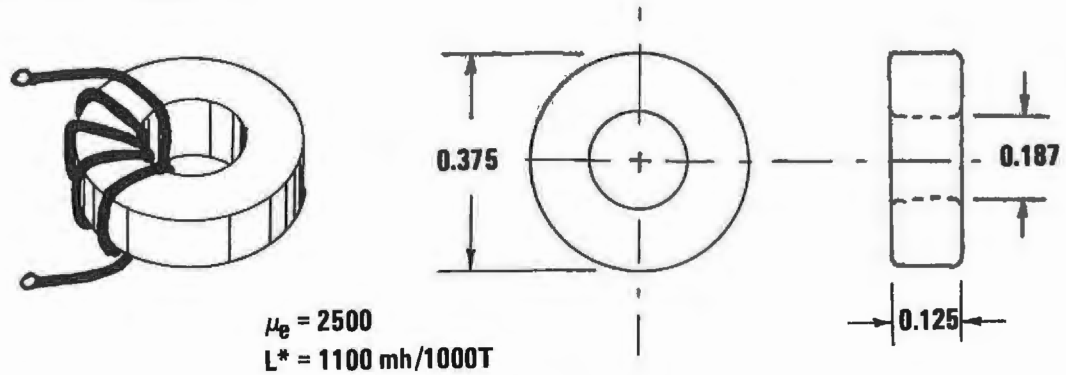
STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Load magnetic core parameters			
	a) for dimensions in inches			
	i) relative permeability of core	μ	[ENT↑]	
	ii) effective core length	ℓ_c	[ENT↑]	
	iii) effective core cross-sectional area	A_c	[A]	μ
	b) for dimensions in centimeters			
	i) relative permeability of core	μ	[ENT↑]	
	ii) effective core length	ℓ_c	[ENT↑]	
	iii) effective core cross-sectional area	A_c	[f] [A]	μ
3	Load air gap length (if L* is to be used, skip this step)			
	a) for dimensions in inches	ℓ_{air}	[B]	ℓ_{air}, cm
	b) for dimensions in centimeters	ℓ_{air}	[f] [B]	ℓ_{air}, cm
4	Load L* (mh/1000T) if air gap is unknown	L*	[f] [D]	ℓ_{air}, cm
5	To calculate the number of turns to achieve a given inductance	L, h	[C]	N
6	To calculate the inductance given the number of turns	N	[f] [C]	L, h
7	Load dc coil current	I_{dc}	[D]	H_{dc}, Oe
				B_{dc}, G
8	If sinewave ac coil excitation is present			
	a) load the rms voltage	E_{rms}, V	[ENT↑]	
	b) load the frequency	f, Hz	[E]	H_{ac} pk, Oe
				H_{dc}, Oe
				H_{total}, Oe
				B_{ac} pk, G
				B_{dc}, G
				B_{total}, G

User Instructions

INDUCTOR DESIGN - MAGNETICS
CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
9	If square-wave coil excitation is present			
	a) load the peak voltage (see Fig. 3-1.1)	E_{pk}	[ENT↑]	
	b) load the frequency	f, Hz	[f] [E]	H_{ac} pk, Oe
				H_{dc}, Oe
				H_{total}, Oe
				B_{ac} pk, G
				B_{dc}, G
				B_{total}, G
10	To obtain the wire size and winding resistance for the above winding, load Program 3-2.			

Example 3-1.1



Design an inductor to have an inductance of 20 millihenries using the above core (a Ferroxcube 266CT125 3B7). The operating frequency is 10 kHz, and the applied ac voltage is 1 Vrms sinewave. There will be 1 mA of dc flowing in the winding.

The core physical constants are needed first:

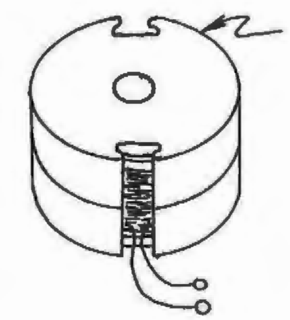
$A = (.125)(.375 - .187)/2 = 11.8 \times 10^{-3}$ inches^2
$\ell_c = \pi(.375 + .187)/2 = .883$ inches (mfgr says .852 in)
$\ell_{air} = 0$ (no air gap)

These dimensions along with $\mu_e = 2500$ are loaded using the A & B keys.

2500.	ENT↑	μ_e
.852	ENT↑	ℓ_c, inches
11.8-03	GSBa	A , inches^2
0.	GSBE	ℓ_{air}
1100.	GSBb	L* (mh/1000T)
4.062-06	***	ℓ_{air} calculated, cm
.020	GSBC	L required, h
134.8+00	***	N calculated (use 135T)
1.-03	GSBD	Idc, amps
77.93-03	***	Hdc, oersteds
194.8+00	***	Bdc, gauss
1.	ENT↑	Vrms
10000.	GSBE	freq, Hz, sinewave
87.71-03	***	Hac peak, oersteds
77.93-03	***	Hdc "
165.6-03	***	H total "
219.3+00	***	Bac peak, gauss
194.8+00	***	Bdc "
414.1+00	***	B total "

Since the core saturates at around 2500 gauss, and this design only excites the core to 414 gauss peak, the design appears adequate from a magnetics standpoint.

Example 3-1.2



Ferrite pot core: Ferroxcube 2213C A400 3B7

ℓ_c = 3.15 cm
A_c = 0.635 cm^2
μ_e = 1845
L* = 400 mh/1000T
B_{max} < 2000 gauss for stable inductance

This pot core is to be used in a tank circuit of a class A tuned amplifier operating at 50 kHz. The dc current is 30 mA, and the applied ac voltage is 10 Vrms. The required inductance is 40 mh (the resonating capacitor is 253 pF). Calculate the effective air gap, the number of turns required, the dc and ac core excitation, and the peak flux density. The following HP-97 printout shows the data entry and calculated parameter output.

1845.	ENT↑	μ_e
3.15	ENT↑	ℓ_c, centimeters
.635	GSBa	A_c, cm^2
400.	GSBd	L* (mh/1000T)
18.24-03	***	ℓ_{air} calculated
.040	GSBC	L required, h
316.2+00	***	N calculated (use 316)
.030	GSBD	Idc, amps
323.9-03	***	Hdc, oersteds
597.6+00	***	Bdc, gauss
10.	ENT↑	Vrms
50000.	GSBE	Freq, Hz, sinewave
12.15-03	***	Hac peak, oersteds
323.9-03	***	Hdc, "
336.1-03	***	H total, "
22.42+00	***	Bac peak, gauss
597.6+00	***	Bdc, "
620.0+00	***	B total, "

A printout of the registers reveals this stored information:

PREG

1.845+03	0	
3.150+00	1	ℓ_c cm
635.0-03	2	A_c, cm^2
18.24-03	3	ℓ_{air}, cm
316.2+00	4	N, turns
50.00+03	5	freq, Hz
323.9-03	6	Hdc, Oe
597.6+00	7	Bdc, gauss
0.000+00	8	
22.42+00	9	Bac pk, gauss
10.00+00	A	Vac, volts
30.00-03	B	Idc, amps
4.000+06	C	L x 10^8
400.0+00	D	L*, mh/1000T
0.000+00	E	
0.000+00	I	

Example 3-1.3

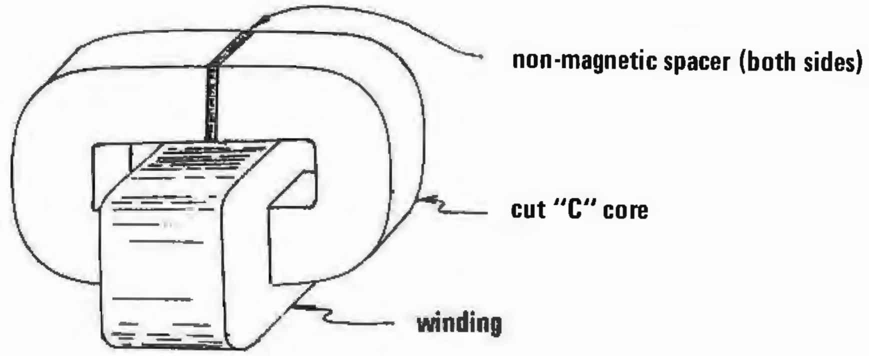


Figure 3-1.2 Inductor on cut C-core.

An inductor to carry dc is needed for the power separation assembly at the end of a coax cable. One ampere dc must flow through the inductor without forcing the B-H loop into a nonlinear region. The inductance needed is 1 henry. Ac signals of 10 Vrms across a frequency band covering 10 Hz to 1000 Hz will be applied in addition to the dc current. A tentative selection is a cut "C" core (see Fig. 3-1.2) with dimensions $A_c = 1.0$ in^2, $\ell_c = 6$ inches, and $\mu = 1000$ (silectron transformer steel). To ensure linear inductance, the peak flux level in the core should not exceed 10000 gauss.

1000.	ENT↑	μ
6.	ENT↑	ℓ_c, inches
1.	GSBA	A_c, inches^2
.062	GSBB	ℓ_{air}, inches (.031 each side)
1.	GSBC	L, h, required
1.460+03	***	N, # turns calc
1.	GSBD	Idc, amps
10.62+00	***	Hdc, oersteds
10.62+03	***	Bdc, gauss
10.	ENT↑	Vrms
10.	GSBE	freq, Hz, sinewave
2.390+00	***	Hac peak, oersteds
10.62+00	***	Hdc, "
13.01+00	***	H total, "
2.390+03	***	Bac peak, gauss
10.62+03	***	Bdc "
13.01+03	***	B total, "

B total exceeds 10000 gauss, use a thicker spacer (larger air gap).

.125	GSBB	new air gap, inches (0.625" each side)
	GSBC	recalculate N
2.026+03	***	N, # turns
	GSBD	recalc, H, Bdc
7.651+00	***	Hdc, oersteds
7.651+03	***	Bdc, gauss
	GSBE	recalc H, Bac, etc.
1.722+00	***	Hac peak, oersteds
7.651+00	***	Hdc, "
9.373+00	***	H total, "
1.722+03	***	Bac peak, guass
7.651+03	***	Bdc, "
9.373+03	***	B total, "

B total is less than 10000 gauss, magnetic design is complete.

3-1 **Program Lisiing I**

001	*LBLa	LOAD CORE PARAMS, CM UNITS
002	STO2	store core area
003	R↓	
004	STO1	store core length
005	R↓	
006	STO0	store core permeability
007	F2?	test for initialization
008	GSB5	
009	GTO4	goto spc, CF3 and rtn
010	*LBLA	LOAD CORE PARAMS, IN. UNITS
011	F2?	test for initialization
012	GSB5	
013	RCLI	convert area in in^2 to cm^2 and store
014	X^2	
015	x	
016	STO2	
017	R↓	
018	RCLI	convert core length in in. to cm and store
019	x	
020	STO1	
021	R↓	
022	STO0	store core permeability
023	GTO4	goto spc, CF3 and rtn
024	*LBLB	LOAD AIR GAP LENGTH, INCHES
025	F2?	test for initialization
026	GSB5	
027	RCLI	convert air gap length to cm
028	x	
029	*LBLb	LOAD AIR GAP LENGTH, CM
030	F2?	test for initialization
031	GSB5	
032	STO3	store air gap length in cm
033	GTO4	goto spc, CF3, and rtn
034	*LBLC	LOAD INDUCTANCE REQUIRED
035	F3?	if no numeric input, jump
036	F.?	
037	GTO0	
038	RCLE	calculate and store $L\cdot10^8$
039	x	
040	STOC	
041	*LBL0	
042	RCLC	calculate and store the number of turns required by using Eq. (3-1.14)
043	GSB6	
044	x	
045	RCL2	
046	÷	$N = \left\{\frac{L(\ell_c + \mu \ell_{air}) \cdot 10^8}{0.4\pi \mu A_c}\right\}^{1/2}$
047	RCL0	
048	÷	
049	√X	
050	STO4	
051	GTO3	print number of turns
052	*LBLc	LOAD NUMBER OF TURNS
053	F3?	if numeric input, store value
054	STO4	
055	RCL4	calculate and store $L\cdot10^8$:
056	X^2	
057	GSB6	
058	÷	
059	RCL2	$L\cdot10^8 = \frac{0.4\pi N^2 \mu A}{\ell_c + \mu \ell_{air}}$
060	x	
061	RCL0	
062	x	
063	STOC	
064	RCLE	divide by 10^8
065	÷	
066	GTO3	goto print subroutine
067	*LBLD	LOAD I_{dc}
068	F3?	if numeric input, store value
069	STOB	
070	RCLB	calculate and print H_{dc}:
071	GSB6	
072	÷	
073	RCL4	
074	x	$H_{dc} = \frac{0.4\pi NI}{\ell_c + \mu \ell_{air}}$
075	PRTX	
076	STO6	
077	RCL0	calculate and store B_{dc}
078	x	B_{dc} $\mu \cdot H_{dc}$
079	STO7	
080	GTO3	goto print subroutine
081	*LBLd	LOAD L*, MH PER 1000 TURNS
082	STOD	
083	1/X	calculate and store equivalent air gap:
084	4	
085	x	
086	Pi	
087	x	
088	RCL2	$\ell_{air} = \frac{4\pi A}{L^*} - \frac{\ell_c}{\mu}$
089	x	
090	RCL1	
091	RCL0	
092	÷	
093	-	
094	STO3	
095	GTO3	goto print subroutine

REGISTERS

0 μ	1 ℓ_c	2 A_c	3 ℓ_{air}	4 N	5 f	6 H_{dc}	7 B_{dc}	8	9 $B_{ac,pk}$
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A V_{ac}	B I_{dc}	C $L \times 10^8$	D L^*	E 10^8	I 2.54

Program Listing II

NOTE FLAG SET STATUS

```
096  *LBLE   LOAD E_rms, f_Hz; CALC H,B
097   F3?
098   F3?    jump if no numeric entry
099   GTO1
100   STO5   store frequency
101   X⇄Y
102   STOA   store rms voltage
103  *LBL1   setup for B_peak calculation
104   RCLA
105      2
106     √X   k = √2 π
107     Pi
108      x
109   GTO2   goto B calculation
110  *LBLe   LOAD E_pk, f_Hz; calc H, B
111   F3?
112   F3?    jump if no numeric entry
113   GTO1
114   STO5   store frequency
115   X⇄Y
116   STOA   store peak voltage
117  *LBL1   setup for B calculation
118   RCLA   k = 4
119      4
120  *LBL2   common B calculation routine
121      ÷
122   RCL5
123      ÷
124   RCL2
125      ÷   B_peak = 10^8 · E / (k A N f)
126   RCL4
127      ÷
128   RCLE
129      x
130   STO9   store B_ac, pk
131   RCL0   calculate and print H_ac, pk
132      ÷   H = B/μ
133   PRTX
134   RCL6   recall and print H_dc
135   PRTX
136      +
137   PRTX   calc and print H_total
138    SPC
139   RCL9   recall and print B_ac, pk
140   PRTX
141   RCL7   recall and print B_dc
142   PRTX
143      +   calculate B_total
144  *LBL3   print and space subroutine
145   PRTX
146  *LBL4   space and CF3 subroutine
147    SPC
148    CF3
149    RTN
150  *LBL5   initialization subroutine
151    EEX
152      8   generate and store 10^8
153   STOE
154     R↓   recover x register
155      2
156      .
157      5   generate and store 2.54
158      4
159   STOI
160     R↓   recover x register
161    RTN   return to main program
162  *LBL6   common magnetics subroutine
163   RCL3
164   RCL0
165      x   calculate:
166   RCL1
167      +   (l_e + μ l_air) / (0.4 π)
168     Pi
169      ÷
170      .
171      4
172      ÷
173    RTN   return to main program
```

NOTE:
To change from the "print" mode to the "R/S" mode for output, change the "print" statements to "R/S" statements at the following line numbers: 075, 133, 135, 137, 140, 142, and 145.

NOTE FLAG SET STATUS

LABELS					FLAGS
A $\mu \uparrow l_c \uparrow A_c$ [in]	B l_{air} [in]	C L → N	D $I_{dc} \to H_{dc}, B_{dc}$	E $V_{rms} \uparrow f_{Hz} \to H_{pk}, B_{pk}$	0
a $\mu \uparrow l_c \uparrow A_c$ [cm]	b l_{air} [cm]	c N → L	d Load L*	e $V_{pk} \uparrow f_{Hz} \to H_{pk}, B_{pk}$	1
0 loop c destination	1 dc flux output routine	2 ac flux output routine	3 print, space, CF3, rtn	4 space CF3, rtn	2 store constants
5 initialize constants	6 $\frac{l_c + \mu_c l_{air}}{0.4\pi}$	7	8	9	3 data entry

SET STATUS				
FLAGS	ON	OFF	TRIG	DISP
0			DEG	FIX
1			GRAD	SCI
2	■		RAD	ENG ■
3		■		n 3

PROGRAM 3-2 FERROMAGNETIC CORE INDUCTOR DESIGN – WIRE SIZE.

Program Description and Equations Used

This program is a companion program to Program 3-1. Given the window area and the number of turns (stored by companion program), this program will calculate the wire size with heavy insulation (class 2) that will fill the window area. If the length of the mean turn is known, the program will also calculate the winding resistance.

The program is also designed to provide information on the wire diamter over class 2 insulation and wire resistance in ohms/inch given the wire size in AWG. The program will also calculate the AWG given the wire diameter over class 2 insulation.

The operation of the program centers around the logarithmic relationship between AWG and the wire cross-sectional area. This logarithmic relationship is:

$$\text{AWG} = \frac{1}{b} \ln \frac{\text{diameter in inches}}{a} \qquad (3\text{-}2.1)$$

where $a' = 0.3245574964$, $b' = -.1159489227$ } bare wire

$a = 0.3137250775$, $b = -.1097881513$ } wire with class 2 insulation

If the total area for a winding of N turns is known, then the area for one turn may be calculated. If the wire is assumed to just fit inside a square with the wire diameter tangent to the sides of the square, then the waste space due to wire stacking can be accommodated (see Fig. 3-6.2). The wire diameter becomes the square root of the square's area. The program uses this algorithm. Once the wire diameter is found, the AWG can be calculated using the logarithmic relationships. The constants for heavy insulation are used. The AWG that is used and is output is the upward rounded value of (1.5 + calculated AWG).

The wire resistance per unit length is inversely proportional to the copper cross-section, hence, the wire size in AWG also bears a logarithmic relationship to the wire resistance. When the wire resistance is desired as a function of the wire AWG, the relationship becomes exponential:

$$R/\ell \text{ (ohms/inch)} = c \cdot e^{(d \cdot AWG)} \qquad (3\text{-}2.2)$$

where $\left.\begin{array}{l} c = 8.371747114 \times 10^{-6} \\ d = -.2317635483 \end{array}\right\}$ annealed copper wire

This exponential relationship is used in conjunction with the mean turn length and the number of turns to calculate the total resistance of the winding. The window area and mean turn length may be entered in either units of inches or centimeters. Centimeter dimensions are converted to inch dimensions before storage within the program.

If the AWG is known and the overall wire diameter including the heavy insulation is desired, Eq. (3-2.1) can be rearranged to yield:

$$\text{diameter in inches} = a \cdot e^{(b \cdot AWG)} \qquad (3\text{-}2.3)$$

This equation is evaluated under label e.

User Instructions

INDUCTOR DESIGN - WIRE SIZE AND RESISTANCE				
window area, cm^2	mean turn length, cm	change # of turns	AWG → ohms/inch	wire diam. → AWG
window area, in^2	mean turn length, in	calculate AWG	calculate winding R	AWG → wire diameter

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Enter window area available for winding			
	a) for dimensions in square inches	A_w, in^2	A	
	b) for dimensions in square centimeters	A_w, cm^2	f A	
3	Enter the length of a mean turn (a turn through the middle of the winding)			
	a) for dimensions in inches	ℓ_t, in	B	
	b) for dimensions in centimeters	ℓ_t, cm	f B	
4	To change the number of turns, or to enter the number of turns if the previous program was not run	N	f C	
5	Calculate the wire AWG that will fill the window area. Heavy insulation (class 2) is assumed.		C	AWG
6	Calculate the winding resistance in ohms		D	R, ohms
7	To find the AWG given the wire diameter over heavy insulation in inches	D_w	f E	AWG
8	To find the wire diameter over heavy insulation given AWG	AWG	E	D_w, in
9	To find the wire resistance per inch of annealed copper wire given AWG	AWG	f D	ohms/in

Example 3-2.1

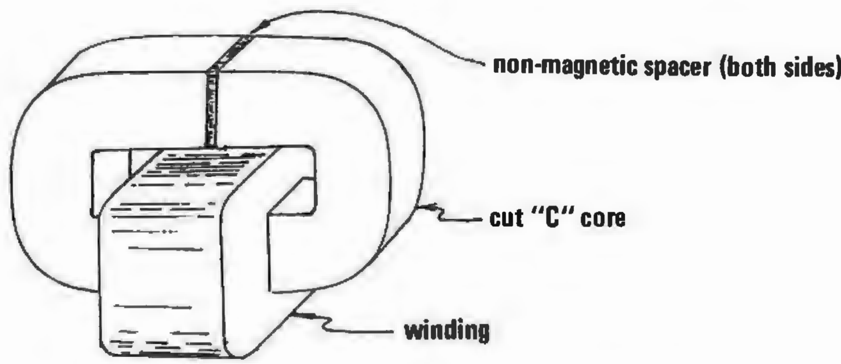


Figure 3-2.1 Inductor on cut C-core.

The inductor in Fig. 3-2.1 was designed to carry dc in Example 3-1.3. If the winding window area is 2 square inches, and the mean turn length is 6 inches, what wire size will fill the winding window, and what will be the total winding resistance?

```
       2. GSBA   window area in square inches
       6. GSBB   mean turn length in inches
          GSBC   start wire size calculation
      22.  ***   wire size in AWG

          GSBD   start winding resistance calculation
16.67+00   ***   winding resistance in ohms
```

3-2

Program Listing I

```
001  *LBLa   LOAD WINDOW AREA IN cm²
002    F0?   test for initialization
003   GSB2
004   RCLI
005     X²   convert area to inches²
006      ÷
007  *LBLA   LOAD WINDOW AREA IN in²
008   STOA   store window area
009    F0?   test for initialization
010   GSB2
011    RTN   return control to keyboard
012  *LBLb   LOAD MEAN TURN LENGTH IN cm
013    F0?   test for initialization
014   GSB2
015   RCLI   convert length to inches
016      ÷
017  *LBLB   LOAD MEAN TURN LENGTH IN in
018   STOC   store mean turn length
019    F0?   test for initialization
020   GSB2
021    RTN   return control to keyboard
022  *LBLc   LOAD NUMBER OF TURNS CHANGE
023   STO4   store new number of turns
024    RTN
025  *LBLC   CALCULATE WIRE AWG
026   RCLA   calculate wire diameter:
027   RCL4      d = √(A_window / n)
028      ÷
029     √X
030   GSB6   calculate AWG from wire diam
031    EEX   using Eq. (3-2.1)
032      +
033   STOB
034   GTO4   goto print, space & dsp subr
035  *LBLD   CALCULATE WINDING RESISTANCE
036   RCLB   use Eq. (3-2.2) to calc
037   GSB5   ohms/inch
038   RCLC
039      x   multiply by total winding
040   RCL4   length to get total
041      x   resistance
042   GTO4   print resistance
043  *LBLd   CONVERT AWG TO OHMS/INCH
044   GSB5   perform conversion
045   GTO4   print result
046  *LBLe   CONVERT WIRE DIAMETER TO AWG
047   GSB6   perform conversion
048   GTO4   print result
049  *LBLE   CONVERT AWG TO WIRE DIAMETER
050   GSB0   interchange registers
051   RCL1
052      x   use Eq. (3-2.3) for
053     eˣ   conversion
054   RCL0
055      x
056   GSB1   interchange registers
057  *LBL4   print, spc, & eng3 subr
058   PRTX
059    SPC
060    ENG
061   DSP3
062    RTN
063  *LBL5   AWG to ohms/inch subroutine
064   GSB0   interchange registers
065   RCL3
066      x   use Eq. (3-2.2) for
067     eˣ   conversion
068   RCL2
069      x
070   GTO1   test for register interchg
071  *LBL6   wire diameter to AWG subr
072   GSB0   interchange registers
073   RCL0
074      ÷   use Eq. (3-2.1) for
075     LN   conversion
076   RCL1
077      ÷
078      .
079      5
080      +
081    FIX
082   DSP0
083    RND
084   GTO1   interchange registers
085  *LBL0   register interchange subr
086    F0?   test for initialization
087   GSB2
088    P⇄S
089    SF2
090    RTN
```

NOTE:
To change from the "print" to "R/S" mode for output, change the "print" statement at line 058 to a "R/S" statemdnt.

REGISTERS

0	1	2	3	4	5	6	7	8	9
				N					
S0 .3137250775	S1 -.1097881513	S2 ×10⁻⁶ 8.371747114	S3 .231765483	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
Window Area, in²	AWG	Mean Turn, in			2.54

Program Listing II

NOTE FLAG SET STATUS

```
091  *LBL2   initialization subroutine
092    ENG   set engr 3 format
093   DSP3
094    F2?   test if P⇄S needed
095    P⇄S
096    P⇄S   execute and note P⇄S
097    SF2
098    CF0   indicate initialization done
099      .
100      3
101      1
102      3
103      7
104      2   .3137250775 → S0
105      5
106      0
107      7
108      7
109      5
110   STO0
111    CLX
112      .
113      1
114      0
115      9
116      7
117      8   -.1097881513 → S1
118      8
119      1
120      5
121      1
122      3
123    CHS
124   STO1
125    CLX
126      8
127      .
128      3
129      7
130      1
131      7
132      4   8.371747114 x 10^-6 → S2
133      7
134      1
135      1
136      4
137    EEX
138    CHS
139      6
140   STO2
141    CLX
142      .
143      2
144      3
145      1
146      7
147      6   .231765483 → S3
148      3
149      5
150      4
151      8
152      3
153   STO3
154    CLX
155      2
156      .   2.54 → RI
157      5
158      4
159   STOI
160     R↓   restore x register
161  *LBL1   subroutine to interchange
162    F2?   registers if flag 2 is set
163    P⇄S
164    RTN
```

NOTE FLAG SET STATUS

LABELS					FLAGS
A load window area in in²	B load mean turn in inches	C calculate AWG	D calculate winding R	E AWG → wire diam	0 store constants
a load window area in cm²	b Load mean turn in cm	c	d	e wire diam → AWG	1
0 P⇄S SF2	1 P⇄S if F2	2 Constant storage	3	4 Print & space	2 P⇄S used
5 AWG → Ω/in	6 wire diam → AWG	7	8	9	3

SET STATUS		
FLAGS (ON / OFF)	TRIG	DISP
0 ON ■	DEG	FIX ■
1	GRAD	SCI
2 OFF ■	RAD	ENG
3		n 0

PROGRAM 3-3 TRANSFORMER LEAKAGE INDUCTANCE AND WINDING CAPACITANCES.

Program Description and Equations Used

This program will calculate the leakage inductance and winding capacitances of a two winding transformer. Both the interwinding capacitance and winding self-capacitances are calculated. The output for both the leakage inductance and winding capacitances are reflected to the primary winding.

Leakage inductance. The total magnetic flux in a transformer is composed of the mutual flux and the leakage flux. The mutual flux follows the core path and links both primary and secondary windings, and results in the mutual, or open-circuit inductance of the transformer. The leakage flux is the relatively small flux which originates in the primary winding and does not link the secondary winding, or vice-versa, and results in the leakage inductance. The leakage flux will be less as the primary and secondary windings are interleaved up to the limit imposed by the space occupied by the insulation between windings. To a degree, the interleaving process is self-defeating, as too much interleaving generates much nonconductive space, and most of the leakage flux flows therein.

Of the many formulas that have been derived for the calculation of leakage inductance, the one by Fortescue [25] is generally accurate and errs, if at all, on the conservative side:

$$L_{leak} = 10.6 \times 10^{-9} \frac{N^2 \cdot MT(2nc + a)}{n^2 b} \qquad (3\text{-}3.1)$$

where

L_{leak} = leakage inductance in henries, referred to the winding having N turns (the primary in this program)

MT = mean-turn length in inches for the whole coil (both windings)

n = number of dielectrics between windings

a = winding buildup in inches

b = winding traverse in inches

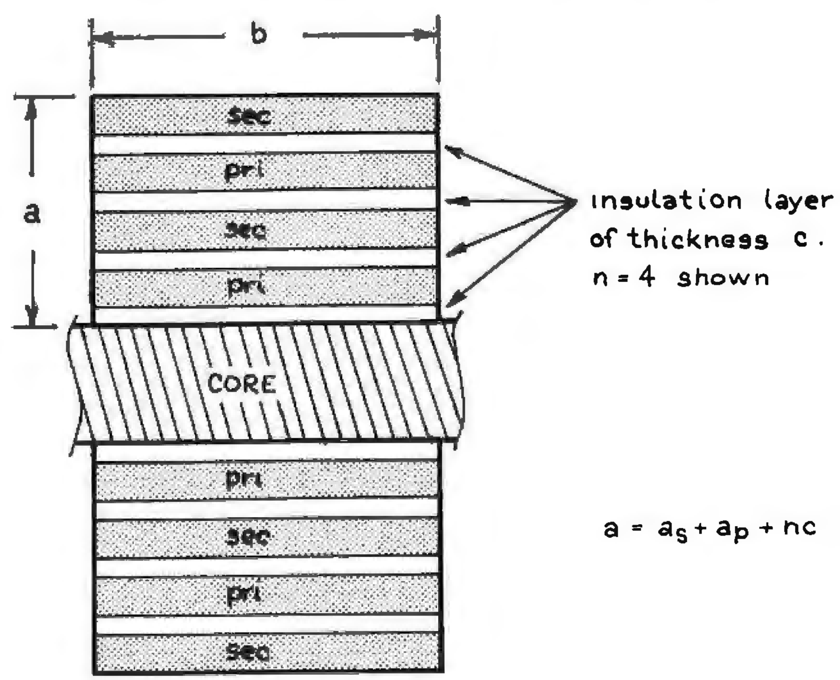
c = dielectric thickness between windings in inches

Interleaving provides the greatest reduction in leakage inductance when the dielectric height, c, is small compared to the window height. When *nc* is comparable to the window height, the leakage inductance does not decrease substantially as the number of interleaves, n, is increased. The lowest leakage inductance will be obtained with a transformer having a small number of turns, a short mean turn length, and a low, wide winding window.

The term "a" in Eq. (3-3.1) refers to the total winding buildup composed of the primary buildup, the secondary buildup, and the insulation layers buildup. If a_p represents the buildup of all the primary interleaves, and a_s represents the buildup of all the secondary interleaves, then:

$$2nc + a = 3nc + a_p + a_s \qquad (3\text{-}3.2)$$

The basis for Eq. (3-3.2) may be seen from Fig. 3-3.1.



$$a = a_s + a_p + nc \qquad (3\text{-}3.3)$$

Figure 3-3.1 Cross-section of transformer winding on a core leg.

<u>Interwinding capacitance</u>. The interwinding capacitance is the primary-secondary capacitance. This capacitance is calculated by considering

the primary and secondary windings as single conducting sheets separated by the dielectric formed by the insulating layer and wire insulation. The capacitance of two flat plates separated by a dielectric is:

$$C = .225 \times 10^{-12}\ \varepsilon \frac{A}{t} \qquad (3\text{-}3.4)$$

where

ε is the relative dielectric constant of the dielectric

A is the area of one plate in inches²

t is the dielectric thickness in inches

For the transformer

$$A = n \cdot MT \cdot b \qquad (3\text{-}3.5)$$

and

$$t = c + t_{\text{primary wire insulation}} + t_{\text{secondary wire insulation}} \qquad (3\text{-}3.6)$$

The wire insulation thickness for heavy insulation (heavy formvar, etc.) can be obtained from the wire AWG. The AWG is obtained from the wire diameter over class 2 insulation by using Eq. (3-2.1), where the wire diameter is calculated by assuming the wire plus insulation just fits in a box as shown by Fig. 3-6.2. The wire diameter over the insulation then becomes:

$$t_{\text{wire, primary}} = \sqrt{\frac{a_p \cdot b}{N_p}} \qquad (3\text{-}3.7)$$

and

$$t_{\text{wire, secondary}} = \sqrt{\frac{a_s \cdot b}{N_s}} \qquad (3\text{-}3.8)$$

The diameter of the bare wire is obtained from AWG by using Eq. (3-2.3). Hence, the thickness of the wire insulation is:

$$t_{\text{wire insulation}} = \tfrac{1}{2}\left(t_{\text{wire + insulation}} - t_{\text{wire}}\right) \qquad (3\text{-}3.9)$$

The wire insulation thickness calculations are performed in the subroutine under label 6 in the HP-67/97 program.

<u>Winding self-capacitance</u>. In a multilayer winding, the voltage between layers is zero at one end of the layer, and $2E/N_L$ at the other where

E is the total winding voltage, and N_L is the number of layers. This voltage gradient model serves as the basis for the total winding capacity as given by Reuben Lee [36].

$$C_i = 1.333 \frac{C_{L_i}}{N_{L_i}} \left\{1 - \frac{1}{N_{L_i}}\right\} \qquad (3\text{-}3.10)$$

i = pri or sec

C_{L_i} is the layer-to-layer capacitance, and is found from Eq. (3-3.4) where

$$A = MT \cdot b \qquad (3\text{-}3.11)$$

and

$$t = t_d + 2t_{\text{wire insulation}} \qquad (3\text{-}3.12)$$

The basis of Eqs. (3-3.11) and (3-3.12) are shown by Fig. 3-3.2.

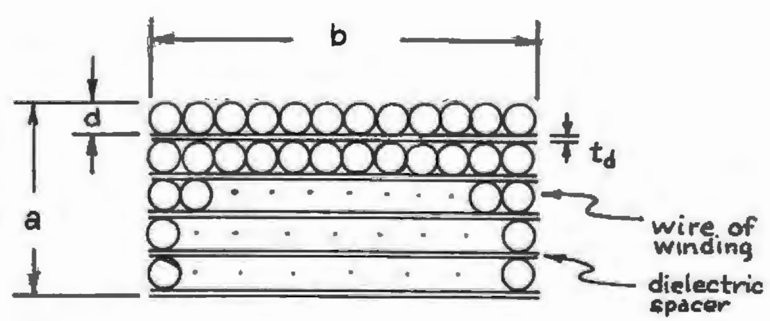


Figure 3-3.2 Cross-section of a winding showing dimensioning.

The number of layers is needed for Eq. (3-3.10) and is found from the number of turns, the interwinding dielectric thickness, and the winding dimensions. The wire cross-sectional area (per Fig. 3-6.2) and the dielectric cross-sectional area must equal the total area available for that winding, i.e.,

$$N_L(d + t_d) = a \qquad (3\text{-}3.13)$$

$$\text{volume} = a \cdot b = \underbrace{N_L \cdot t_d \cdot b}_{\text{spacer volume}} + \underbrace{N \cdot d^2}_{\text{wire volume}} \qquad (3\text{-}3.14)$$

Substituting Eq. (3-3.13) into (3-3.14) and solving for N_L yields:

$$N_{L_i} = \frac{N_i\, d_i}{b_i} \qquad (3\text{-}3.15)$$

where d is the quadratic solution to:

$$N_i d_i^{\,2} + \left(N_i t_{d_i}\right) d_i - a_i b_i = 0 \qquad (3\text{-}3.16)$$

i = pri or sec

The program calculates the secondary winding capacity and reflects it to the primary winding:

$$C_{sec}@\ \text{primary} = C_{sec} \cdot \left(\frac{N_s}{N_p}\right)^2 \qquad (3\text{-}3.17)$$

The total winding capacity seen at the primary is the sum of the reflected secondary winding capacitance, and the primary winding capacitance.

TRANSFORMER LEAKAGE L AND WINDING C				
winding traverse	$N_p \uparrow N_s$	# of dielectrics	print?	Calculate $L_{leakage}$ $C_{winding}$ $C_{interwinding}$
pri ↑ sec buildup	average mean turn length	$t_{d_p} \uparrow t_{d_s} \uparrow t_d$	$\epsilon_{pri} \uparrow \epsilon_{sec} \uparrow \epsilon_{sp}$	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
	(note flag status)			
2	Load both sides of data card			
3	Select print or R/S option using toggle		[f] [D]	0, R/S
			[f] [D]	1, print
			[f] [D]	0, R/S
				⋮
4	Load winding traverse in inches	b, in	[f] [A]	
5	Load winding buildup in inches:			
	a) primary buildup	a_p, in	[ENT↑]	
	b) secondary buildup	a_s, in	[A]	
6	Load number of turns:			
	a) primary turns	N_p	[ENT↑]	
			[f] [B]	
7	load average mean-turn length for the whole			
	transformer winding in inches	MT, in	[B]	
8	load the number of dielectrics	n	[f] [C]	
9	load dielectric thickness in inches:			
	a) primary interwinding dielectric	t_{d_p}, in	[ENT↑]	
	b) secondary interwinding dielectric	t_{d_s}, in	[ENT↑]	
	c) primary-secondary dielectric	t_d, in	[C]	
10	Load relative dielectric constants:			
	a) average for primary interwinding	ϵ_{pri}	[ENT↑]	
	dielectric and wire insulation			
	b) average for secondary interwinding	ϵ_{sec}	[ENT↑]	
	dielectric and wire insulation			
	c) primary-secondary spacer	ϵ_{sp}	[D]	

TRANSFORMER LEAKAGE L AND WINDING C
CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
11	Calculate L's and C's		[E]	
	Primary leakage inductance			L_{leak}, h
				space
	Secondary wire AWG			AWG, sec
	Number of secondary winding layers			# layers
	Secondary winding C reflected to primary, F			$C_{sec@pri}$
				space
	Primary wire AWG			AWG, pri
	Number of primary layers			# layers
	Primary winding capacity in farads			$C_{primary}$
	Total winding capacity reflected to primary			C_{total}
				space
	primary-secondary interwinding capacitance, F			$C_{pri-sec}$
12	Data review:			
	Go back and key any entry key without			
	keying in any numeric entry to view the			
	presently stored variable. See Example			
	3-3.1.			

Example 3-3.1

Find the primary leakage inductance and winding capacitances of a transformer having the following specifications:

traverse: 2"

number of pri-sec dielectrics: 4 (3 interleaves)

dielectric thickness: 0.050"

pri-sec insulator dielectric constant: 10

mean turn length for whole transformer: 5"

Primary

number of turns: 100

buildup: 0.25"

interwinding dielectric thickness: 0.002"

average interwinding dielectric and wire insulation dielectric constant: 10

Secondary

number of turns: 1000

buildup: 0.3"

interwinding dielectric thickness: 0

average interwinding dielectric and wire insulation dielectric constant: 5

HP printout for Example 3-3.1

```
      2.  GSBa   winding traverse

    .25  ENT↑   primary winding buildup
     .3  GSBA   secondary winding buildup

   100.  ENT↑   primary winding turns
  1000.  GSBb   secondary winding turns

     5.  GSBB   mean turn length for whole transformer

     4.  GSBc   number of pri-sec dielectrics

   .002  ENT↑   primary interwinding dielectric thickness
     0.  ENT↑   secondary interwinding dielectric thickness
    .05  GSBC   pri-sec dielectric thickness

    10.  ENT↑   average primary dielectric constant
     5.  ENT↑   average secondary dielectric constant
    10.  GSBD   pri-sec dielectric, dielectric constant


         GSBE   calculate L's and C's
19.05-06  ***   primary leakage inductance, henrys

24.00+00  ***   secondary wire AWG
24.49+00  ***   number of secondary layers
23.19-09  ***   secondary interwinding C seen @ primary, F

14.00+00  ***   primary wire AWG
6.972+00  ***   number of primary layers
678.6-12  ***   primary interwinding capacity, F
23.87-09  ***   total interwinding capacity @ primary, F

1.699-09  ***   pri-sec winding capacity, F
```

Data Review printout for Example 3-3.1

```
            GSBa
 2.000+00   ***   traverse

            GSBA
300.0-03    ***   secondary winding buildup
250.0-03    ***   primary winding buildup

            GSBb
 1.000+03   ***   secondary turns
100.0+00    ***   primary turns

            GSBB
 5.000+00   ***   mean turn length

            GSBc
 4.000+00   ***   number of dielectrics

            GSBC
50.00-03    ***   primary-sec dielectric thickness
 0.000+00   ***   secondary interwinding dielectric thickness
 2.000-03   ***   primary interwinding dielectric thickness

            GSBD
10.00+00    ***   pri-sec dielectric, dielectric constant
 5.000+00   ***   secondary average dielectric constant
10.00+00    ***   primary average dielectric constant
```

3-3

Program Listing I

```
001  *LBLA   I/O PRIMARY & SEC BUILDUP
002    EEX
003   GSB0
004      0
005   GTO1
006  *LBLa   I/O WINDING TRAVERSE (b)
007      2
008   GTO1
009  *LBLB   I/O AVERAGE MEAN TURN (MT)
010      3
011   GTO1
012  *LBLb   I/O PRIMARY AND SEC TURNS
013      5
014   GSB0
015      4
016   GTO1
017  *LBLC   I/O DIELECTRIC THICKNESSES
018      8   I/O pri-sec spacer thickness
019   GSB0
020      7   I/O secondary intrawinding
021   GSB0   dielectric thickness
022      6   I/O primary intrawinding
023   GTO1   dielectric thickness
024  *LBLc   I/O NUMBER OF DIELECTRICS
025      9
026   GTO1
027  *LBLD   I/O DIELECTRIC CONSTANTS
028      1   I/O dielectric constant
029      2   of pri-sec spacer
030   GSB0
031      1   I/O secondary insulation
032      1   dielectric constant
033   GSB0
034    EEX   I/O primary insulation
035      1   dielectric constant
036  *LBL1   subroutine to I/O last item
037   GSB0
038   GTO8
039  *LBL0   main I/O subroutine
040   STOI   store index
041     R↓   recover entry
042    F3?   if flag 3, set flag 1
043    SF1
044    F1?   if flag 1, store entry
045   STOi
046    F1?   if flag 1, recover
047     R↓   previous entry
048    F1?   if flag 1, return
049    RTN
050   RCLi   recall and print item
051   GTO7
052  *LBLd   PRINT-R/S TOGGLE
053    F0?
054   GTO2
055    SF0
056    EEX
057   GTO8
058  *LBL2
059    CF0
060    CLX
061   GTO8
062  *LBLE   CALCULATE L's & C's
063    P⇄S   calculate leakage inductance
064   RCL4
065    P⇄S
066   RCL4
067   RCL9
068      ÷
069     X²
070      x
071   RCL3
072      x
073   RCL9
074   RCL8
075      x
076      3
077      x
078   RCL0
079      +
080   RCL1
081      +
082      x
083   RCL2
084      ÷
085   GSB3
086   RCL1   calculate and store
087   RCL2   2·t_wire, secondary
088      x
089   RCL5
090      ÷
091   RCL7
092   GSB6
093   STOB
094     R↓   recover d/b
095   RCL5   recall Ns
096   GSB5   calc secondary capacitance:
097    P⇄S
098   RCL1
099    P⇄S
100   RCLB
101   RCL7
102      +
103   GSB4
104      x
105   RCL5   reflect secondary capaci-
106   RCL4   tance to primary:
107      ÷
108     X²
109      x
110   STOC
111   GSB3
112   RCL0
```

$$L_{leak} = \frac{10.6\ N_p^2 \cdot MT \cdot (3nc + a_p + a_s)}{10^{+9}\ n^2\ b}$$

REGISTERS

0 a_p, pri buildup	1 secondary buildup	2 b, winding traverse	3 MT, mean turn length	4 N_p	5 N_s	6 $t_{D_{pri}}$	7 $t_{D_{sec}}$	8 C, $t_{D_{spacer}}$	9 n, the # of dielectrics
S0 ϵ_{pri}	S1 ϵ_{sec}	S2 ϵ_{spacer}	S3 225×10^{-15}	S4 10.6×10^{-9}	S6 k_1 = .3137250775	S6 k_1' = .3245574964	S7 k_2 = -.1097881513	S8 k_2' = -0.1159489227	S9

A 2x primary wire insulation thickness	B 2x secondary wire insulation thickness	C Csec, or dsec	D	E 1.33333···	I index or scratchpad

Program Listing II

NOTE FLAG SET STATUS

```
113  RCL2
114  x       calculate and store
115  RCL4    2·t_wire, primary
116  ÷
117  RCL6
118  GSB6
119  STOA
120  R↓      calc primary capacitance
121  RCL4
122  GSB5
123  P⇄S
124  RCL0
125  P⇄S
126  RCLA
127  RCL6
128  +
129  GSB4
130  x
131  GSB7
132  RCLC    calculate and print:
133  +         C_pri - N²·C_sec
134  GSB3
135  RCL9    calculate interwinding cap.:
136  P⇄S
137  RCL2
138  P⇄S
139  x
140  RCLA
141  RCLB
142  +
143  2
144  ÷
145  RCL8
146  +
147  GSB4
148  *LBL3   print or R/S subroutine
149  GSB7
150  GTO8
151  *LBL4   capacity subroutine
152  ÷
153  RCL3    MT
154  x
155  RCL2    b
156  x
157  P⇄S
158  RCL3    .225 x 10^-12
159  P⇄S
160  x
161  RTN
162  *LBL5   intrawinding capacity subr
163  x       calc and print # of layers
164  GSB7
165  1/X     calculate winding capacity
166  ENT↑    term:
167  ENT↑
168  EEX
169  X⇄Y
170  -
171  x
172  RCLE
173  x
174  RTN
175  *LBL6   wire AWG and insulation thk.
176  2       calculate wire diameter:
177  ÷
178  STOI
179  X²
180  +
181  √X
182  RCLI
183  -
184  STOI
185  RCL2    calculate d/b
186  ÷
187  RCLI    calculate insulation thick.
188  RCLI    calculate wire AWG
189  P⇄S
190  RCL5    k1
191  ÷
192  LN
193  RCL7    k2
194  ÷
195  ENT↑
196  ENT↑
197  EEX     calculate and print
198  +       integral wire size
199  INT
200  GSB7
201  R↓      calculate bare wire
202  RCL8    diameter from AWG
203  y
204  e^x
205  RCL6
206  P⇄S
207  x
208  -       calculate 2·t_insulation
209  RTN
210  *LBL7   print or R/S subroutine
211  F0?
212  PRTX
213  F0?
214  RTN
215  R/S
216  RTN
217  *LBL8   space and clear flag 3 subr
218  F0?
219  SPC
220  CF1
221  CF3
222  RTN
223  GTO8
```

NOTE FLAG SET STATUS

LABELS					FLAGS
A I/O of pri & sec buildup	B I/O of mean turn length	C I/O of dielectric thick	D I/O of relative dielectric const	E calculate L & C's	0 print
a I/O winding traverse	b I/O turns N_p, N_s	c I/O of # of dielectrics	d print or R/S toggle	e	1 input
0 I/O subroutine	1 I/O subroutine	2 print or R/S toggle	3 print or R/S and space	4 capacitance subroutine	2
5 winding C subroutine	6 wire diameter subroutine	7 print or R/S subroutine	8 space & CF1,3 subr	9	3 input

SET STATUS			
FLAGS	ON	OFF	TRIG / DISP
0	■		DEG / FIX
1		■	GRAD / SCI
2			RAD / ENG ■
3		■	n 3

PROGRAM 3-4 STRAIGHT WIRE AND LOOP WIRE INDUCTANCE.

Program Description and Equations Used

This program calculates the inductance of straight wire lengths and single square wire loops. The permeability of the wire is taken into account only for the inductance calculation, but not for skin depth; therefore, the inductance calculated is the low frequency inductance.

The calculation of wire inductance can be an important design parameter in some instances. For example, the bonding wire inductance of high speed, wideband hybrid integrated circuits affects circuit performance. Wire self-inductance is also important in the design of high frequency (1000 Hz), high power (megawatt) power conversion equipment such as SCR inverters, choppers, cycloconverters, and phase delay rectifiers.

The inductance of a straight wire increases with permeability and length, and decreases with increasing diameter. The combined effect of permeability, length, and diameter is not described simply, but can be easily solved with a scientific calculator. For example, the inductance of copper wire is strongly influenced by diameter while the inductance of a high permeability wire such as permalloy is relatively unaffected by diameter.

The formulas used herein come from Grover [30], and can also be found in Terman [52]. Two basic formulas are used, one for straight wire, and another for wire loops. These formulas are algebraically manipulated to obtain expressions for each of the four variables; wire diameter (d), wire length (ℓ), relative permeability (μ), and inductance in μh (L). The program works in the units of centimeters, but the user may input data in either inch or centimeter units.

Figure 3-4.1 shows the definitions of the wire terms.

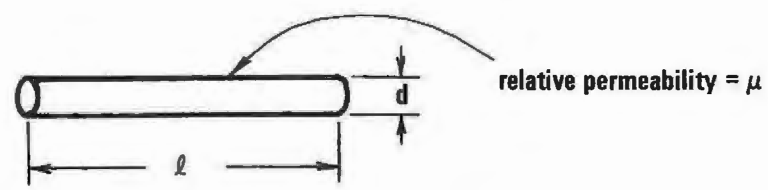


Figure 3-4.1 Straight wire terms.

The formulas for the straight wire case are:

$$L = (2 \times 10^{-3})\, \ell \left\{ \ell n\left(\frac{4\ell}{d}\right) + \frac{\mu}{4} - 1 \right\},\ \mu h \tag{3-4.1}$$

$$d = \frac{4\ell}{e^{(L/(2\ell \times 10^{-3}) - \mu/4 + 1)}} \tag{3-4.2}$$

$$\mu = 4\left\{ \frac{L}{2\ell \times 10^{-3}} + 1 - \ell n\left(\frac{4\ell}{d}\right) \right\} \tag{3-4.3}$$

To obtain the wire length, a Newton-Raphson iterative solution is employed (see Program 1-3 for details), because the equation for ℓ has a logarithm containing ℓ.

$$\ell = \frac{L}{(2 \times 10^{-3})\left\{ \ell n\left(\frac{4\ell}{d}\right) + \frac{\mu}{4} - 1 \right\}} \tag{3-4.4}$$

The Newton-Raphson solution finds where a function is zero, therefore, let:

$$f(\ell) = \ell - \frac{L}{(2 \times 10^{-3})\left\{ \ell n\left(\frac{4\ell}{d}\right) + \frac{\mu}{4} - 1 \right\}} = 0 \tag{3-4.5}$$

and

$$f'(\ell) = \frac{df(\ell)}{d\ell} = 1 + \frac{L}{(2 \times 10^{-3}\ell)\left\{ \ell n\left(\frac{4\ell}{d}\right) + \frac{\mu}{4} - 1 \right\}} \tag{3-4.6}$$

The initial guess for ℓ is 1, and the ℓ value for each succeeding iteration is given by:

$$\ell_{i+1} = \ell_i - \frac{f(\ell_i)}{f'(\ell_i)} \tag{3-4.7}$$

The iteration is terminated when:

$$\left| \ell_{i+1} - \ell_i \right| < 10^{-6} \tag{3-4.8}$$

Figure 3-4.2 shows the definitions of the loop wire terms.

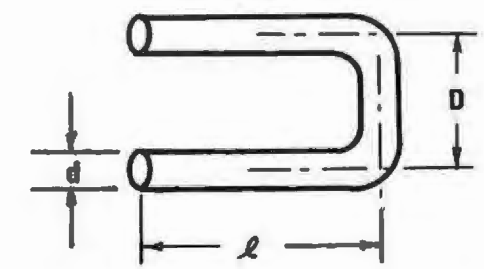


Figure 3-4.2 Loop wire terms.

The formulas for the loop wire case are:

$$L = (4 \times 10^{-3}\, \ell) \left\{ \ell n\left(\frac{2D}{d}\right) + \frac{\mu}{4} - \frac{D}{\ell} \right\},\ \mu h \tag{3-4.9}$$

$$d = \frac{2D}{e^{(L/(4 \times 10^{-3}\ell) - \mu/4 + D/\ell)}} \tag{3-4.10}$$

$$\ell = \frac{\frac{L}{4 \times 10^{-3}} + D}{\ell n\left(\frac{2D}{d}\right) + \frac{\mu}{4}} \tag{3-4.11}$$

$$\mu = 4\left\{ \frac{L}{4 \times 10^{-3}\ell} + \frac{D}{\ell} - \ell n\left(\frac{2D}{d}\right) \right\} \tag{3-4.12}$$

Keys "a" through "d" set up the dimension units to be used for input or output (inches or centimeters), and the configuration (straight wire or loop wire). When the loop wire configuration is selected (key "c"), the loop separation, D, must also be entered via key "c."

Keys "A" through "D" provide the program input/output functions. Use of these keys following numeric input signals an input to the program. Use of these keys without numeric entry, or following the clear key (E) signals an output is required from the program. Flag 3 is used to indicate input or output within the program.

3-4 User Instructions

STRAIGHT WIRE AND LOOP WIRE INDUCTANCE				
centimeter units	inch units	wire loop, enter D	straight wire	
wire diam d	wire length ℓ	permeablty μ	inductance L	clear input mode

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of data card			
2	Select dimension units			
	a) centimeter units		[f] [A]	1.000
	b) inch units		[f] [B]	2.540
3	Select configuration			
	a) wire loop, load loop separation	D	[f] [C]	
	b) straight wire		[f] [D]	
4	To calculate wire diameter, d			
	a) load wire length	ℓ	[B]	
	b) load wire permeability	μ	[C]	
	c) load required inductance	L, μh	[D]	
	d) start solution		[A]	d
5	To calculate wire length, ℓ			
	a) load wire diameter	d	[A]	
	b) load wire permeability	μ	[C]	
	c) load required inductance	L, μh	[D]	
	d) start solution execution		[B]	ℓ
6	To calculate permeability, μ			
	a) load wire diameter	d	[A]	
	b) load wire length	ℓ	[B]	
	c) load required inductance	L, μh	[D]	
	d) start solution execution		[C]	μ
7	To calculate inductance, L			
	a) load wire diameter	d	[A]	
	b) load wire length	ℓ	[B]	
	c) load permeability	μ	[C]	
	d) start solution execution		[D]	L, μh
8	To clear input mode (reset flag 3)		[E]	

Example 3-4.1

Find the inductance of a straight gold wire 0.001 inch in diameter and 0.3 inch long (a hybrid integrated circuit interconnect wire).

```
         GSBb   set inches
         GSBd   set straight wire mode
   .001  GSBA   load wire diameter in inches
   .300  GSBB   load wire length in inches
  1.000  GSBC   load wire relative permeability
         GSBD   calculate inductance
  0.010  ***    inductance in microhenries
```

Example 3-4.2

Find the length of a 4/0 copper cable (.528 in diam) having an inductance of 6 microhenries

```
         GSBb   set inches
         GSBd   set straight wire mode
   .528  GSBA   load wire diameter
  1.000  GSBC   load relative permeability of wire
  6.000  GSBD   load required inductance
         GSBB   calculate wire length*
182.258  ***    length, inches

 12.000  ÷
 15.188  ***    length, feet
```

*Computation time takes about 1 minute.

Example. 3-4.3

A pair of 4/0 wires run 20 feet between a capacitor module and an inverter module in an ac traction motor controller. The wire separation is twice the wire diameter. What parasitic inductance does the wire add in series with the capacitors? 4/0 wire is .528 inches in diameter.

```
           GSBb   set inch mode
    .528   ENT↑  }
              +  }   calculate and enter the wire separation,
   1.056    ***  }   and select wire loop configuration
           GSBc
    .528   GSBA   load wire diameter in inches
  20.000   ENT↑  }
  12.000      ×  }   calculate and load wire length in inches
 240.000    ***  }
           GSBB
   1.000   GSBC   load permeability of wire
           GSBD   calculate inductance of wire loop
   3.973    ***   inductance, microhenries
```

If the maximum parasitic inductance that can be tolerated is 2 microhenries, how long can the feeder wires be if the other parameters don't change?

```
   2.000   GSBD   load required inductance in μh
           GSBE   calculate loop length
 120.948    ***   loop length, inches

  12.000      ÷
  10.079    ***   loop length, feet
```


Program Listing I

Step	Key	Comment
001	*LBLa	SET CM UNIT MODE
002	EEX	
003	STO9	store cm → cm conversion
004	RTN	
005	*LBLb	SET INCH UNIT MODE
006	2	
007	.	
008	5	store in → cm conversion
009	4	
010	STO9	
011	RTN	
012	*LBLc	LOAD WIRE LOOP SEPARATION
013	SF0	indicate wire loop mode
014	CF3	
015	4	goto data entry subroutine
016	GTO0	
017	*LBLd	SET STRAIGHT WIRE MODE
018	CF0	indicate straight wire mode
019	RTN	
020	*LBLA	I/O OF WIRE DIAMETER, d
021	2	
022	EEX	
023	CHS	store 0.002
024	3	
025	STO8	
026	R↓	recover input
027	0	
028	F3?	if numeric input,
029	GTO0	goto data input subroutine
030	F0?	jump if wire loop mode
031	GTO1	
032	RCL1	calculate and store d for straight wire case:
033	4	
034	×	
035	GSB6	
036	+	$d = \dfrac{4\ell}{e^{\left(\frac{L}{2\ell\times10^{-3}} - \frac{\mu}{4} + 1\right)}}$
037	RCL2	
038	−	
039	e^x	
040	÷	
041	STO0	
042	GTO7	goto unit conversion & print
043	*LBL1	calculate and store d for loop wire case
044	GSB8	
045	GSB6	
046	×	
047	2	
048	÷	
049	−	
050	e^x	$d = \dfrac{2D}{e^{\left(\frac{L}{4\ell\times10^{-3}} - \frac{\mu}{4} + \frac{D}{\ell}\right)}}$
051	RCL4	
052	×	
053	ENT↑	
054	+	
055	STO0	
056	GTO7	goto unit conv and print
057	*LBLB	I/O OF WIRE LENGTH, ℓ
058	EEX	
059	F3?	if numeric input,
060	GTO0	goto input subroutine
061	F0?	jump if loop wire mode
062	GTO1	
063	STO1	store "1" for initial guess
064	*LBL4	Newton-Raphson loop start
065	RCL1	
066	4	
067	×	calculate and store $f(\ell)$:
068	RCL0	
069	÷	
070	LN	
071	EEX	$\ln\left\{\frac{4\ell}{d}\right\} + \frac{\mu}{4} - 1$
072	−	
073	RCL2	
074	+	
075	F2?	test for subroutine exit
076	RTN	
077	STOE	finish $f(\ell)$ calculation
078	RCL8	
079	×	
080	1/X	
081	RCL5	$f(\ell) = \ell - \dfrac{L}{2\times10^{-3}\left\{\ln\left(\frac{4\ell}{d}\right) + \frac{\mu}{4} - 1\right\}}$
082	×	
083	CHS	
084	RCL1	
085	+	
086	STO7	
087	RCL5	calculate and apply $f'(\ell)$:
088	RCL8	
089	RCL1	
090	×	
091	RCLE	$f'(\ell) = 1 + \dfrac{L}{(2\times10^{-3}\ell)\left\{\ln\left(\frac{4\ell}{d}\right) + \frac{\mu}{4} - 1\right\}^2}$
092	X^2	
093	×	
094	÷	
095	EEX	
096	+	
097	ST÷7	calculate correction
098	RCL7	apply correction
099	ST−1	
100	ABS	
101	EEX	
102	CHS	test for loop exit
103	6	
104	X≤Y?	
105	GTO4	
106	RCL1	recall and print
107	GTO7	
108	*LBL1	calculate ℓ for loop wire case
109	RCL5	
110	RCL8	

REGISTERS

0 diameter cm	1 length cm	2 $\mu/4$	3	4 wire separation, cm	5 inductance L	6	7 scratch	8 2×10^{-3}	9 1 or 2.54
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E scratch	I index

Program Listing II

```
111  ENT↑
112  +
113  ÷
114  RCL4
115  +
116  RCL4      ℓ = (L/(4×10^-3) + D) / (ln(2D/d) + μ/4)
117  ENT↑
118  +
119  RCL0
120  ÷
121  LN
122  RCL2
123  +
124  ÷
125  STO1      store ℓ
126  *LBL7     unit conversion & prt subr
127  RCL9      recall unit conversion
128  ÷
129  *LBL2     print and space subroutine
130  PRTX  --  can be R/S statement
131  SPC
132  RTN
133  *LBLC     I/O OF PERMEABILITY, μ
134  4
135  ÷
136  RCL9      undo unit conversion
137  ÷
138  2         if numeric input, goto
139  F3?       data input subroutine
140  GTO0
141  F0?       jump if wire loop mode
142  GTO3
143  GSB6      start calculation for
144  +         straight wire
145  RCL1
146  4
147  GTO1
148  *LBL3     loop wire   calculation
149  GSB6
150  2         L/(4ℓ×10^-3)
151  ÷
152  RCL4
153  RCL1      D/ℓ
154  ÷
155  +
156  RCL4      D
157  2
158  *LBL1     common calculation routine
159  x
160  RCL0
161  ÷
162  LN
163  -
164  STO2      store μ/4
165  4
166  x         calculate μ
167  GTO2      goto print and space subr
168  *LBLD     I/O OF INDUCTANCE, L
169  RCL9      undo unit conversion
170  ÷
171  5         if numeric input, goto
172  F3?       data input subroutine
173  GTO0
174  F0?       jump if loop wire mode
175  GTO3
176  SF2       calc: ln(4ℓ/d) + μ/4 - 1
177  GSB4
178  GTO1      jump
179  *LBL3     calculate:
180  RCL4
181  ENT↑
182  +
183  RCL0      2{ln(2D/d) + μ/4 - D/ℓ}
184  ÷
185  LN
186  GSB8
187  +
188  ENT↑
189  +
190  *LBL1     common inductance calculation
191  RCL8      x(2ℓ × 10^-3)
192  x
193  RCL1
194  x
195  STO5      store inductance
196  GTO2      goto print and space subr
197  *LBL0     data input subroutine
198  STOI      store register index
199  R↓        recover input
200  RCL9      apply unit conversion and
201  x         store entry
202  STOi
203  RTN       return to main program
204  *LBL6     subroutine to calculate:
205  EEX
206  RCL5
207  RCL8      L/(2ℓ × 10^-3)
208  ÷
209  RCL1
210  ÷
211  RTN
212  *LBL8     subroutine to calculate:
213  RCL2
214  RCL4
215  RCL1      μ/4 - D/d
216  ÷
217  -
218  RTN
219  *LBLE     CLEAR INPUT MODE
220  CF3
221  RTN
```

LABELS					FLAGS	SET STATUS		
A d	B ℓ	C μ	D L	E clear input	0 wire loop	FLAGS	TRIG	DISP
a cm units	b inch units	c wire loop	d straight wire	e	1	ON OFF 0 ■ 1 2 ■ 3 ■	DEG GRAD RAD	FIX ■ SCI ENG n 3
0 data entry	1 used	2 output routine w/o unit conv	3 partial calc of loop wire μ	4 Newton - Raphson Loop	2 subr 4 exit			
5	6 calc L/(.002 ℓ)	7 output routine w/ unit conv	8 calc of μ/4 - D/ℓ	9 calc of ln(4ℓ/d)+μ/4 -1	3 data entry			

PROGRAM 3-5 AIR-CORE SINGLE-LAYER INDUCTOR DESIGN.

Program Description and Equations Used

This program uses Wheeler's equation [55] to solve for the various parameters relating to single-layer, air-core inductor design. The basic form of Wheeler's equation is:

$$L(\mu h) = \frac{a^2 n^2}{9a + 10\ell} \quad \text{(use inch dimensions)} \qquad (3\text{-}5.1)$$

This equation provides answers within 1% accuracy for all values of $2a/\ell$ less than 3, and the results will be about 4% low when $2a/\ell = 5$ (short coils).

There are five parameters that can be used to describe an air-core inductor: the coil radius in inches (a), the coil length in inches (ℓ), the number of turns (n), the winding pitch ($p = \ell/n$), and the inductance in microhenries (L). Of this set of five parameters, only four are independent since ℓ, n, and p are interrelated; hence, given any three independent parameters, the fourth independent parameter, and the remaining dependent parameter can be found. For example, L can be calculated given a, ℓ, and n, or a, n, and p.

Wheeler's equation may be algebraically manipulated to yield the other independent variables.

Solving for ℓ given a, n, and L:

$$\ell = \frac{a^2 n^2 - 9a\,L}{10\,L} \qquad (3\text{-}5.2)$$

Solving for ℓ given n and p:

$$\ell = n \cdot p \qquad (3\text{-}5.3)$$

Solving for n given a, ℓ, and L:

$$n = \frac{1}{a}\sqrt{L(9a + 10\ell)} \qquad (3\text{-}5.4)$$

Solving for n given a, p, and L: find quadratic solution of

$$a^2n^2 - 10Lpn - 9aL = 0 \qquad (3\text{-}5.5)$$

Solving for p given ℓ and n:

$$p = \ell/n \qquad (3\text{-}5.6)$$

Solving for p given a, n, and L:

$$p = \frac{1}{10n}\left\{\frac{a^2n^2}{L} - 9a\right\} \qquad (3\text{-}5.7)$$

Solving for L given a, n, and p:

$$L = \frac{a^2n^2}{9a + 10np} \qquad (3\text{-}5.8)$$

Solving for a given ℓ, n, and L: find quadratic solution of

$$n^2a^2 - 9La = 10\ell L = 0 \qquad (3\text{-}5.9)$$

The program uses these equations as follows. The appropriate input keys are assumed to have been executed prior to an output request. Label "A" inputs or outputs the coil radius in inches, a. The input is stored in R0, and Eq. (3-5.9) is used for output.

Label "B" inputs or outputs the number of turns, n. The input is stored in R1, and if p was previously entered, ℓ is calculated using Eq. (3-5.3). For output, Eq. (3-5.5) is used if p, ℓ, and a are specified, otherwise, Eq. (3-5.4) is used.

Label "C" inputs or outputs the coil length, ℓ. For input, the coil length is stored in R2, flag 0 is cleared, and a new p is calculated and stored using Eq. (3-5.6). For output, if p has been previously entered, use Eq. (3-5.3), otherwise use Eq. (3-5.2).

Label "D" inputs or outputs the winding pitch, p. For input, the new pitch is stored in R3, flag 0 is set, and new ℓ is calculated with Eq. (3-5.3). For output, Eq. (3-5.6) is used.

Label "E" inputs or outputs the coil inductance, L, in microhenries. For input, the value is stored in R4. For output, Eq. (3-5.1) is used, and the new inductance value stored.

Label "c" calculates the wire diameter given the wire AWG with heavy insulation. The wire diameter over heavy insulation bears an

exponential relationship to the wire gauge:

$$\text{Diameter (inches)} = k_1 \cdot e^{k_2 \cdot AWG} \qquad (3\text{-}5.10)$$

where $k_1 = 0.31373$
and $k_2 = -.109788$

On the first execution of this routine, the constants k_1 and k_2 are stored into R8 and R9 respectively. Flag 2 is initially set after magnetic card reading to indicate constant storage required, and is reset upon test.

Label "d" calculates the AWG of the wire given the diameter over the insulation in inches:

$$AWG = \frac{1}{k_2} \cdot \ell n \left\{\frac{\text{Diameter}}{k_1}\right\} \qquad (3\text{-}5.11)$$

Label "e" is used to clear flag 3 to indicate data output desired.

Keys "A" through "E" leave flag 3 cleared after the associated routine finishes, i.e., data output mode is set unless further numeric entry is made.

The routines under keys "d" and "e" do not alter the state of flag 3. For example, one may load the wire AWG, use key "c" to convert to wire diameter, and then use key "D" to load this value as the winding pitch (close wound coils).

Highest coil Q's are generally obtained when the space between the wires equals the wire diameter (pitch equals twice the wire diameter). Callendar's equation [13] can be used to estimate the Q of a coil with this pitch:

$$Q = \frac{\sqrt{\text{freq in Hz}}}{\frac{2.71}{a} + \frac{2.13}{\ell}} \quad \text{(use inch dimensions)} \qquad (3\text{-}5.12)$$

For RF coils where the skin depth is less than the wire diameter, Callendar's equation is accurate to within a few percent. For close wound coils, the calculated Q will be high by a factor of 1.9.

HP-67 users may want to make the following program changes to make the final number in the display unambiguous. For example, label "C" causes both the number of turns and the coil length to be printed with the coil length being displayed last. To change the program so only the number of turns is displayed and printed, change lines 122 through 126 of the program as follows:

```
122  RCL3
123    x
124  STO2
125  RCL1
126  GTO8
```


User Instructions

AIR-CORE, SINGLE-LAYER, SOLENOIDAL INDUCTOR DESIGN				
		AWG → diam	diam → AWG	clear input mode
coil radius a, in	# of turns n	coil length ℓ, in	pitch $p = \ell/n$	inductance L, uh

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Select problem type:			
	a) to find L & p given a, n, & ℓ			
	i) load the coil radius	a, in	A	
	ii) load the number of turns	n	B	
	iii) load the coil length	ℓ, in	C	
	iv) calculate the coil inductance		E	L, µh
	v) calculate the winding pitch		D	p, in/T
	b) to find L & ℓ given a, n, & p			
	i) load the coil radius	a, in	A	
	ii) load the number of turns	n	B	
	iii) load the winding pitch	p, in/T	D	
	iv) calculate the coil inductance		E	L, µh
	v) calculate the coil length		C	ℓ, in
	c) to find n & p given a, ℓ, & L			
	i) load the coil radius	a, in	A	
	ii) load a dummy value for n *	1	B	
	iii) load the winding length	ℓ, in	C	
	iv) load desired inductance	L, µh	E	
	v) calculate the # of turns and		B	n, turns
	the winding pitch			p, in/T
	d) to find n & ℓ given a, n, & L			
	i) load the coil radius	a, in	A	
	ii) load the winding pitch	p, in/T	D	
	iii) load desired inductance	L, µh	E	
	iv) calculate the number of turns		B	n, turns
	and the winding length			ℓ, in
	e) to find ℓ & p given a, n, & L			
	i) load the coil radius	a, in	A	
	ii) load the desired number of turns	n	B	
	iii) load the desired inductance	L, µh	E	
	iv) calculate the inductor length **		C	ℓ, in
	v) calculate the winding pitch		D	p, in/T

3-5 **User Instructions**

AIR-CORE, SINGLE-LAYER, SOLENOIDAL INDUCTOR DESIGN

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
2	f) to find a & ℓ given n, p, & L			
	i) load the desired number of turns	n	[B]	
	ii) load the desired winding pitch	p, in/T	[D]	
	iii) load the desired inductance	L, μh	[E]	
	iv) calculate the coil radius		[A]	a, in
	v) calculate the coil length		[C]	ℓ, in
	g) to find a & p given n, ℓ, & L			
	i) load the desired number of turns	n	[B]	
	ii) load the desired coil length	ℓ, in	[C]	
	iii) load the desired inductance	L, μh	[E]	
	iv) calculate the coil radius		[A]	a, in
	v) calculate the winding pitch		[D]	p, in/T
3	Go back to any part of step 2, or stop			
4	To convert wire AWG to diameter over heavy (class 2) insulation	AWG	[f] [C]	diam, in
5	To convert wire diameter over heavy insulation to AWG	diam, in	[f] [D]	AWG
6	To clear input mode, i.e., to request output after numeric operations have been performed from the keyboard		[f] [E]	
	Notes:			
	* p = ℓ/n, a non-zero n is required for proper program operation. The dummy n is replaced with the calculated n under label B.			
	** A negative value for the inductor length means the required inductance cannot be realized with the chosen radius and number of turns. Either increase n or a.			

Example 3-5.1

An air-core coil is to be wound in a ½ inch form using #18 AWG HF wire at a pitch of twice the wire diameter. What number of turns are required for an inductance of 500 nanohenry (0.5 μh), and what will the winding length be?

.250	GSBA	load coil radius in inches
18.000	GSBc	load wire AWG
0.043	***	wire diameter over HF insulation
2.000	×	calculate winding pitch (2 x diam)
	GSBD	load winding pitch
.500	GSBE	load required inductance in microhenry
	GSBB	calculate turns and coil length
8.956	***	number of turns (use 9 turns)
0.776	***	coil length in inches

Example 3-5.2

A 6 turn coil on a 6 inch form is closewound with #4/0 wire. The wire is 0.750 inches over the insulation. What is the coil inductance and length?

3.000	GSBA	load the coil radius in inches
6.000	GSBB	load the number of turns
.750	GSBD	load winding pitch
	GSBE	calculate inductance
4.500	***	inductance in microhenries
	GSBC	calculate coil length
4.500	***	coil length in inches

Program Listing I

```
001  *LBLA   I/O OF COIL RADIUS, a
002   F3?    jump if numeric entry
003   GTO0
004   RCL1   use quadratic equation
005   X²     to find a in:
006   STO5   a²n² - 9aL - 10ℓL = 0
007   9
008   RCL4
009   x
010   CHS
011   STO6
012   RCL2
013   RCL4
014   x
015   EEX
016   1
017   x
018   CHS
019   STO7
020   GSB9   gosub quadratic solution
021   STO0   store a
022   GTO8   goto print & space subr
023  *LBL0   coil radius data input
024   STO0   store coil radius
025   RTN    return control to keyboard
026  *LBLC   I/O OF COIL LENGTH,
027   F3?    jump if numeric entry
028   GTO0
029   F0?    jump if p entered last
030   GTO1
031   RCL0   calculate and store:
032   RCL1
033   x
034   X²
035   RCL4
036   ÷      ℓ = 1/10 (a²n²/L - 9a)
037   RCL0
038   9
039   x
040   -
041   EEX
042   1
043   ÷
044   STO2
045   GTO8   goto print and space subr
046  *LBL0
047   CF0    indicate ℓ entered last
048   STO2   store ℓ
049   RCL1   calculate and store
050   ÷      p = ℓ/n
051   STO3
052   RTN
053  *LBL1   calculate and store
054   RCL1
055   RCL3   ℓ = n(ℓ/n)
056   x
057   STO2
058   GTO8   goto print and space subr
059  *LBLD   I/O OF COIL PITCH, p
060   F3?    jump if numeric entry
061   GTO0
062   RCL2   calculate and store
063   RCL1
064   ÷      p = ℓ/n
065   STO3
066   GTO8   goto print and space subr
067  *LBL0
068   STO3   store p
069   RCL1   calculate and store
070   x      ℓ = p·n
071   STO2
072   SF0    indicate p entered last
073   RTN
074  *LBLB   I/O OF COIL TURNS, n
075   F3?    jump if numeric entry
076   GTO0
077   F0?    jump if p entered last
078   GTO1
079   GSB3   calculate and store n:
080   RCL4
081   x
082   √X
083   RCL0
084   ÷      n = (1/a) √(L(9a - 10ℓ))
085   STO1
086   PRTX
087   1/X
088   RCL2
089   x
090   STO3
091   GTO8
092  *LBL1   calculate and store n from
093   RCL0   quadratic solution to:
094   X²
095   STO5
096   RCL3
097   RCL4
098   x
099   EEX    a²n² - 10Lpn - 9aL = 0
100   1
101   x
102   CHS
103   STO6
104   RCL0
105   RCL4
106   x
```


Program Listing II

NOTE FLAG SET STATUS

```
107   9
108   x
109   CHS
110   STO7
111   GSB9
112   STO1
113   PRTX
114   RCL3
115   x
116   STO2
117   GTO8
118  *LBL0
119   STO1   store number of turns
120   F0?    jump if p entered last
121   GTO0
122   RTN
123  *LBL0   calculate and store new
124   RCL3   coil length
125   x
126   STO2
127   RTN
128  *LBLE   I/O OF INDUCTANCE, L (μh)
129   F3?    jump if numeric entry
130   GTO0
131   RCL0   use Wheeler's equation
132   RCL1   to calculate inductance
133   x      (Eq. (3-5.1)):
134   X²
135   GSB3   L = a²n² / (9a + 10ℓ)
136   ÷
137   STO4
138  *LBL8   print and space subroutine
139   PRTX
140   SPC
141   RTN
142  *LBL0
143   STO4   store inductance input
144   RTN
145  *LBL9   quadratic equation solution
146   RCL5   subroutine.
147   ST÷6
148   ST÷7   If ax² + bx + c = 0
149   2
150   ST÷6   then the positive root is:
151   RCL6
152   CHS    x = - b/2a + √((b/2a)² + c/a)
153   ENT↑
154   X²
155   RCL7
156   -
157   √X
158   +
159   RTN
160  *LBL3   9a + 10ℓ calculation subr
161   RCL0
162   9
163   x
164   RCL2
165   EEX
166   1
167   x
168   +
169   RTN
170  *LBLc   AWG → WIRE DIAMETER
171   F2?    constant initialization
172   GSB2   needed?
173   RCL9
174   x
175   eˣ     diameter = k₁·e^(k₂·AWG)
176   RCL8
177   x
178   GTO8
179  *LBLd   WIRE DIAMETER → AWG
180   F2?    constant initialization
181   GSB2   needed?
182   RCL8
183   ÷
184   LN
185   RCL9   AWG = (1/k₂) ln{diameter/k₁}
186   ÷
187   INT
188   GTO8
189  *LBL2   constant initialization
190   .
191   3
192   1
193   3
194   0
195   4
196   STO8   store k₁
197   R↓     recover x register
198   .
199   1
200   0
201   9
202   7
203   3
204   3
205   CHS
206   STO9   store k₂
207   R↓     recover x register
208   RTN
209  *LBLe   CLEAR INPUT MODE
210   CF3
211   RTN
```

NOTE: Print statements are located at steps 086 and 139 and may be changed to R/S if desired.

NOTE FLAG SET STATUS

REGISTERS

0	1	2	3	4	5	6	7	8	9
a	n	ℓ	p	L	quadratic equation terms: a	b	c	wire AWG constants: k_1	k_2
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I

LABELS

A	B	C	D	E
I/O coil radius	I/O coil length	I/O # of turns	I/O pitch	I/O inductance
a	b	c	d	e
0 local label	1 local label	2 constant storage	3 9a + 10ℓ	4
5	6	7	8 print & space	9 quadratic solution

FLAGS

0	p entered last
1	
2	store coefficients
3	data entry

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	DEG	FIX ■
1			GRAD	SCI
2	■		RAD	ENG
3		■		n 3

PROGRAM 3-6 AIR-CORE MULTILAYER INDUCTOR DESIGN.

Program Description and Equations Used

This program uses a modification of Bunet's formula [11], Eq. (3-6.1), to design air-core, multilayer solenoidal inductors (inch dimensions).

$$L\ (\mu h) = \frac{a^2 n^2}{9a + 10\ell + 8.4c + 3.2\,c\ell/a} \qquad (3\text{-}6.1)$$

The coil dimensions are shown in Fig. 3-6.1, and the range of usefulness of the program can be ascertained from Table 3-6.1.

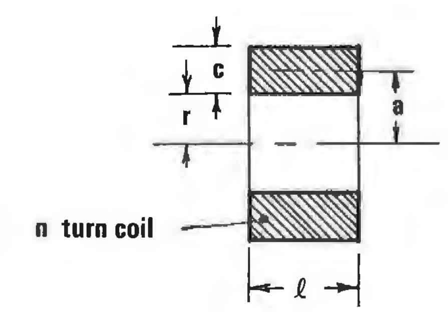


Figure 3-6.1 Multilayer coil dimensions.

Table 3-6.1 Accuracy estimates for Bunet's equation.

c/a ratio	2a/ℓ ratio for 1% accuracy	other accuracies 2a/ℓ	%
1/20	≤ 3	5	4
1/5	≤ 5	10	2
1/2	≤ 2	5	3
1/1	≤ 1.5	5	5

The modification to Eq. (3-6.1) consists of replacing the mid-coil radius, a, by the inner radius, r:

$$a = r + \frac{c}{2} \qquad (3\text{-}6.2)$$

The coil is generally wound on a coil form, hence, r and ℓ are known from the coil form dimensions. The coil mid-radius, a, is dependent upon the coil buildup, and is generally not known at the inception of the design.

If the wire and insulation occupy a box as shown in Fig. 3-6.2,

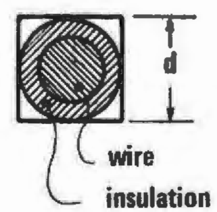


Figure 3-6.2 Wire cross-section.

then the total area occupied by n turns of this wire would be:

$$A_{total} = n \cdot d^2 \qquad (3\text{-}6.3)$$

This area is also expressible in terms of the coil dimensions:

$$A_{total} = c \cdot \ell \qquad (3\text{-}6.4)$$

Hence,

$$n \cdot d^2 = c \cdot \ell \qquad (3\text{-}6.5)$$

or

$$c = \frac{n \cdot d^2}{\ell} \qquad (3\text{-}6.6)$$

A fifth order polynomial in n may be derived to yield the number of turns of wire with diameter d, given the required inductance, L, the coil inner radius, r, and the coil width, ℓ. Taking Eq. (3-6.1), multiplying both sides by the denominator term, and clearing fractions yields:

$$a^3 n^2 - L\{9a^2 + (10\ell + 8.4c)\,a + 3.2c\ell\} = 0 \qquad (3\text{-}6.7)$$

Substituting Eq. (3-6.2) for a, and Eq. (3-6.6) for c, and collecting terms in like powers of n results in the following 5th order polynomial equation:

$$f(n) = An^5 + Bn^4 + Cn^3 + Dn^2 - En - F = 0 \qquad (3\text{-}6.8)$$

$$A = \left(\frac{d^2}{2\ell}\right)^3 \qquad (3\text{-}6.9)$$

$$B = 3r\left(\frac{d^2}{2\ell}\right)^2 \qquad (3\text{-}6.10)$$

$$C = 3r^2\left(\frac{d^2}{2\ell}\right) \qquad (3\text{-}6.11)$$

$$D = r^3 - \left(\frac{d^2}{2\ell}\right)^2 (25.8\ L) \qquad (3\text{-}6.12)$$

$$E = L\left\{\frac{d^2}{2\ell}(34.8r + 10\ell) + 3.2d^2\right\} \qquad (3\text{-}6.13)$$

$$F = rL\,(10\ell + 9r) \qquad (3\text{-}6.14)$$

The Newton-Raphson iterative procedure described in Program 1-3 is used to find the largest positive real root for n in Eq. (3-6.8). If the initial guess for n is larger than the largest root, the method will converge to the largest root when the function is a polynomial as in the present case. An initial guess of 10000 turns is used. If a larger number of turns is expected, the user may want to increase the initial guess which is located at step 084 of the program.

If r, c, ℓ, and L are specified, then the solution for n becomes somewhat simpler. Since r and c are both known, a can be calculated from Eq. (3-6.2). With this calculation, all parameters except n are known in Eq. (3-6.1), and n becomes:

$$n = \frac{1}{a}\left\{L(9a + 10 + c(8.4) + 3.2\ell/a\,)\right\}^{\frac{1}{2}} \qquad (3\text{-}6.15)$$

Once n has been calculated, the wire diameter, d, can be calculated from Eq. (3-6.6) as given below:

$$d = \sqrt{\frac{c\ell}{n}} \qquad (3\text{-}6.16)$$

So far, the two cases for the number of turns have been derived. Likewise, there are two cases for the calculation of L. Given r, ℓ, c, and n, Eqs. (3-6.2) and (3-6.1) may be used to calculate L. If the wire diameter, d, had been specified instead of the coil thickness, c, then

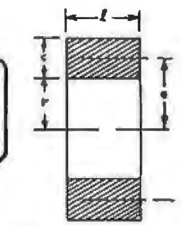
Eqs. (3-6.6) and (3-6.1) are used to calculate L.

Program constants

Since all program steps were used to code the program equations, no room remains for the program constants. These constants are recorded on another magnetic card, and are loaded after the program magnetic card loading. Load the following registers, and record the data on both sides of the data card (2 WDATA commands):

8.4 → R7
3.2 → R8
10. → R9

3-6 **User Instructions**



AIR-CORE MULTILAYER INDUCTOR DESIGN

coil inner radius, r (I)	wire diam, d (I) / winding thickness, c (I)	winding length, ℓ (I)	number of turns, n (I/O)	inductance in μh, L (I/O)

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card and either side of data card			
2	Load inner coil radius	r, in	A	
3	Load wire diameter	d, in	f B	
	or	or		
	Load winding thickness	c, in	B	
4	Load winding width	ℓ, in	C	
5	To calculate inductance in microhenries			
	a) load the number of coil turns	n	D	
	b) calculate inductance		E	L, μh
6	To calculate the number of turns			
	a) load the desired inductance	L, μh	E	
	b) calculate the number of turns		D	n

Example 3-6.1

Find the number of turns of #24 HF wire (0.0224 inches over insulation) to be wound on a bobbin that has a 0.3 inch inner radius and is 0.5 inch wide to obtain an inductance of 200 microhenries. Also find the coil thickness.

```
      .30 GSBA   load bobbin inner radius (in)

    .0224 GSBb   load wire diameter over insulation (in)

      .50 GSBC   load bobbin width (in)

   200.00 GSBE   load inductance required (μh)

          GSBD   calculate # of turns & coil thickness*
   122.66  ***   number of turns (use 123)
   0.1231  ***   coil thickness, inches
```

Example 3-6.2

Calculate the inductance of an 18 turn coil of 4/0 wire with 6 turns per layer wound on a 6 inch diameter form. 4/0 wire is 0.75 inch over the insulation.

```
     3.00 GSBA   load coil inner radius (in)

      .75 GSBb   load wire diameter over insulation (in)

     6.00        calculate coil width:
     4.50  ***   coil turns per layer x thickness per turn
          GSBC   load coil width

    18.00 GSBD   load number of turns

          GSBE   calculate inductance
  50.6345  ***   inductance in microhenries
```

* Requires about a minute to compute.

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD COIL INNER RADIUS
002	STO1	
003	GTO9	
004	*LBLB	LOAD COIL THICKNESS
005	SF0	indicate thickness loaded
006	STO2	store thickness
007	GTO9	goto OF3, space & return
008	*LBLb	LOAD WIRE DIAMETER
009	CF0	indicate wire diam. loaded
010	STO6	store wire diameter
011	GTO9	goto OF3, space & return
012	*LBLC	LOAD WINDING LENGTH
013	STO3	store
014	GTO9	goto OF3, space, & return
015	*LBLD	I/O OF COIL TURNS
016	F3?	if input, jump
017	GTO0	
018	F0?	if coil thickness loaded,
019	GTO1	use other routine
020	RCL6	calculate n given r, ℓ,
021	X²	d, and L
022	RCL3	
023	ENT↑	
024	+	calculate and temporarily
025	÷	store $d^2/(2\ell)$
026	STOI	
027	3	calculate and store n^5 coef
028	Yˣ	$A = \{d^2/(2\ell)\}^3$
029	STOA	
030	RCLI	
031	X²	calculate and store n^4 coef
032	RCL1	
033	3	$B = 3r\left(\frac{d^2}{2\ell}\right)^2$
034	GSB5	
035	STOB	
036	RCLI	calculate and store n^3 coef
037	RCL1	
038	X²	
039	3	$C = 3r^2\left(\frac{d^2}{2\ell}\right)$
040	GSB5	
041	STOC	
042	RCL1	
043	3	
044	Yˣ	calculate and store n^2 coef
045	2	
046	5	
047	.	$D = r^3 - \left(\frac{d^2}{2\ell}\right)^2 (25.8 \cdot L)$
048	8	
049	RCLI	
050	X²	
051	RCL4	
052	GSB5	
053	-	
054	STOD	
055	3	
056	4	
057		
058	8	
059	RCL1	calculate and store n^1 coef
060	x	
061	RCL9	
062	RCL3	
063	GSB4	$E = L\left\{\frac{d^2}{2\ell}(34.8r + 10\ell) + 3.2d^2\right\}$
064	RCLI	
065	x	
066	RCL8	
067	RCL6	
068	X²	
069	GSB4	
070	RCL4	
071	x	
072	STOE	
073	RCL9	
074	RCL3	
075	x	calculate and store n^0 coef
076	9	
077	RCL1	
078	GSB4	
079	RCL4	$F = rL(9r + 10\ell)$
080	RCL1	
081	GSB5	
082	STOI	
083	EEX	setup initial guess for n
084	4	in Newton-Raphson soln
085	STO5	
086	*LBL8	Newton-Raphson start
087	RCL5	
088	ENT↑	
089	ENT↑	
090	ENT↑	
091	RCLA	calculate and store
092	x	
093	RCLB	$f(n_i) = An_i^5 + Bn_i^4 + Cn_i^3 + Dn_i^2 - En_i - F$
094	+	
095	x	
096	RCLC	
097	+	
098	x	
099	RCLD	
100	+	
101	x	
102	RCLE	
103	-	
104	x	
105	RCLI	
106	-	
107	STO2	
108	CLX	calculate
109	RCL5	
110	RCLA	$f'(n_i) = 5An_i^4 + 4Bn_i^3 + 3Cn_i^2 + 2Dn_i - E$
111	5	
112	GSB5	

REGISTERS

0	1	2	3	4	5	6	7	8	9
a	r	c	ℓ	L	n	d	8.4	3.2	10
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
A	B	C	D	E	F

Program Listing II

```
113  RCLB
114     4
115     x
116     +
117     x
118  RCLC
119     3
120     x
121     +
122     x
123  RCLD
124  ENT↑
125     +
126     +
127     x
128  RCLE
129     -
130  ST÷2   calc & store f(n_i)/f'(n_i)
131  RCL2   apply correction:
132  ST-5   n_i+1 = n_i - f(n_i)/f'(n_i)
133   ABS
134     .   test for loop exit
135     1
136  X≤Y?
137  GTO8
138  RCL6
139    X²
140  RCL5   ← can be R/S statement
141  PRTX   print n
142     x
143  RCL3   calculate, print and store
144     ÷   coil thickness, c:
145  STO2   c = nd²/ℓ
146  GTO2
147 *LBL0   input storage routine for
148  STO5   number of turns input
149  GTO9   goto CF3, space and return
150 *LBL1   calculate the number of turns
151  GSB3   given r, ℓ, c, and L
152  RCL4
153     x
154    √X   n = (1/a){L(9a + 10ℓ + 8.4c + 3.2 cℓ/a)}^(1/2)
155  RCL0
156     ÷
157  STO5
158  PRTX ← can be R/S statement
159   1/X
160  RCL2
161  RCL3
162  GSB5
163    √X
164  STO6
165  GTO2
166 *LBLE   I/O OF INDUCTANCE
167  STO4   store inductance entry
168   F3?
169  GTO9   jump if input
170   F0?   if winding thickness loaded,
171  GTO1   skip thickness calculation
172  RCL6
173    X²   calculate and store
174  RCL3   thickness:
175     ÷
176  RCL5   c = nd²/ℓ
177     x
178  STO2
179 *LBL1
180  GSB3
181   1/X   calculate and store
182  RCL0   inductance:
183  RCL5   L = a²n² / (9a + 10ℓ + 8.4c + 3.2 cℓ/a)
184     x
185    X²
186     x
187  STO4
188 *LBL2   print and space subroutine
189  DSP4
190  PRTX ← can be R/S statement
191  DSP2
192 *LBL9   CF3 and space subroutine
193   CF3
194   SPC
195   RTN
196 *LBL3   inductance factor
197  RCL1   calculation subroutine
198  RCL2
199     2   calculate and store:
200     ÷   a = r + c/2
201     +
202  STO0
203     9
204     x
205  RCL3   calculate:
206  RCL9   9a + 10ℓ + 8.4c + 3.2 cℓ/a
207  GSB4
208  RCL7
209  RCL8
210  RCL0
211     ÷
212  RCL3
213  GSB4
214  RCL2
215 *LBL4   x, + subroutine
216     x
217     +
218   RTN
219 *LBL5   x, x subroutine
220     x
221     x
222   RTN
```

LABELS					FLAGS
A load inner radius	B Load winding thickness	C Load winding Length	D I/O number of turns	E I/O inductance	0 set for winding thickness
a	b	c	d	e	1
0 local Loop destination	1 Local Label	2 print & space subroutine	3 inductance subroutine	4 x, + subroutine	2
5 x, x subroutine	6	7	8 turns subroutine	9 space & rtn subroutine	3 data entry

SET STATUS			
FLAGS	ON	OFF	
0		■	
1			
2			
3		■	
TRIG	DEG	GRAD	RAD
DISP	FIX ■	SCI	ENG
n	2		

PROGRAM 3-7 CYLINDRICAL SOLENOID DESIGN.

Program Description and Equations Used

This program provides the coil winding particulars and the coil electrical characteristics given the specifications for a cylindrical solenoid. These specifications are:

1) Minimum plunger attractive force in pounds (F),
2) Initial air gap length in inches (ℓ_{air}),
3) Maximum flux density in the air gap (B_{max}) in gauss,
4) Maximum coil current density in amperes/in^2 (Δ),
5) Maximum coil buildup, or thickness, (w) in inches,
6) Coil excitation voltage (E) in volts, or current (I) in amperes,
7) Optionally, the magnetic path area (A_{iron}) in inches^2, the magnetic path length (ℓ_{iron}) in inches, and the magnetic permeability (μ).

The length of the magnetic path is assumed to be zero unless step 7 is exercised.

The characteristics that the program calculates are:

1) Plunger diameter in inches (D_p),
2) Number of turns in the coil (N),
3) Coil wire AWG using class 2 or heavy insulation,
4) Coil length in inches (ℓ_{coil}),
5) Coil inductance in henries (L),
6) Coil resistance in ohms (R),
7) Coil power dissipation in watts (P),
8) Actual B in the core and in the air-gap, and
9) Actual F.

With the maximum flux density in the air gap and plunger attractive force specified, the area of the air gap can be calculated from:

$$A_{air} = F \cdot k_1 / (B_{air}{}^2) \quad (3\text{-}7.1)$$

where k_1 is the constant of proportionality relating flux density in the air gap to pressure in pounds/in^2

$$k_1 = 1.73 \times 10^6 \qquad (3\text{-}7.2)$$

If the plunger area is assumed equal to the air gap area, the plunger diameter can be calculated using:

$$D_p = 2\cdot\left(A_{air}/\pi\right)^{\frac{1}{2}} \qquad (3\text{-}7.3)$$

Once the plunger diameter is known, then a value for the winding thickness may be loaded into the program. The smallest dimension of the winding should not exceed 3 inches to allow adequate thermal conduction for the heat generated with the coil, thus avoiding high internal coil temperatures. If the program calculates a short coil length, then the thickness is not restrained. A long coil restrains the coil thickness to 3 inches or less. Several iterations of the program solution may be required until satisfactory values for coil length and width (thickness) are found.

Given the excitation voltage, inverse current density in the coil (M) in circular mils per ampere, and the coil dimensions as defined by Fig. 3-7.1, the number of turns required is given by Eq. (3-7.4). The derivation of this equation is given later.

$$N = E\cdot M/(\pi(D_p + w)) \qquad (3\text{-}7.4)$$

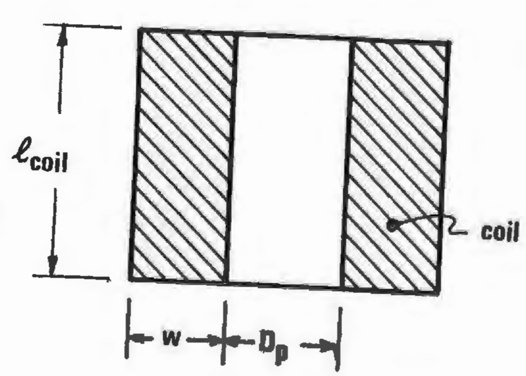


Figure 3-7.1 Solenoid coil dimensions.

If the coil is excited with a current, then the number of turns is:

$$N = (NI)/I \qquad (3\text{-}7.5)$$

where NI is the coil ampere-turns which is calculated from B_{max} later.

The cross-sectional area of the coil $(w\cdot\ell_{coil})$ consists of current carrying wire and noncurrent carrying insulation and space. The shape factor (sf) is the ratio of the current carrying area to the total area of the coil. If the wire plus insulation is assumed to occupy a square with side d as shown in Fig. 3-6.2, and the winding cross-section is occupied by N of these squares, then the shape factor is:

$$sf = \frac{\pi}{4}\left\{\frac{\text{diameter of bare wire}}{d}\right\}^2 \qquad (3\text{-}7.6)$$

The diameters of both the bare wire and the wire with insulation bear exponential relationships to the wire AWG as given by Eq. (3-2.1). Substituting these relationships into Eq. (3-7.6) yields:

$$sf = \frac{\pi}{4}\left\{\frac{a'}{a}\, e^{AWG(b' - b)}\right\}^2 \qquad (3\text{-}7.7)$$

where

$$\frac{\pi}{4}\left\{\frac{a'}{a}\right\}^2 = .8418900745 \qquad (3\text{-}7.8a)$$

$$2(b' - b) = -1.21690938 \times 10^{-2} \qquad (3\text{-}7.8b)$$

The coil has N wires each carrying in current, I; thus the current density in the coil is:

$$\Delta = (NI)/(sf\cdot l_{coil}\cdot w) \qquad (3\text{-}7.9)$$

where Δ is specified by the user through M:

$$k_2 = M\cdot\Delta = (\text{cir-mils/A})(\text{A/in}^2) = (4 \times 10^6)/\pi \qquad (3\text{-}7.10)$$

Solving for the coil length between Eqs. (3-7.9) and (3-7.10) yields:

$$l_{coil} = (NI\cdot M)/(sf\cdot k_2\cdot w) \qquad (3\text{-}7.11)$$

The coil ampere-turns, NI, is calculated from B_{max} using the "Ohm's law" of magnetics:

$$MMF = \phi\cdot\mathcal{R} \qquad (3\text{-}7.12)$$

where ϕ is the flux and is continuous throughout the magnetic and air paths and is analogous to electric current. The reluctance, $\mathcal{R}$, is the magnetic resistance, and the magnetomotive force, MMF, is the magnetic "voltage" source. The total reluctance is the sum of the individual reluctances making up the magnetic circuit and the MMF is proportional to the current in the coil:

$$MMF = 0.4\mu N I \tag{3-7.13a}$$

$$\mathcal{R} = \sum_i \frac{\ell_i}{\mu_i \cdot A_i} \tag{3-7.13b}$$

The electromagnet model used by this program has two sections, the magnetic path, and the air gap. Usually the air gap reluctance is the dominant term. Noting that the relative permeability for air is unity, and

$$\phi = B_{iron} \cdot A_{iron} = B_{max} \cdot A_{air} \tag{3-7.14}$$

then solving Eq. (3-7.12) for NI yields:

$$NI = \frac{B_{max} A_{air}}{A_{iron}} \left\{ \frac{\ell_{iron}}{\mu_{iron}} + \ell_{air} \frac{A_{iron}}{A_{air}} \right\} \frac{k_3}{0.4\pi} \tag{3-7.15}$$

where k_3 = 2.54, the inch to centimeter conversion ratio. The iron area, A_{iron}, refers to the smallest iron area, which may not be next to the air gap.

An iterative method is required to find the wire AWG and coil length. An initial shape factor of 0.5 is assumed, the coil length is obtained using Eq. (3-7.11). The wire diameter over insulation is obtained using

$$d = (w \cdot \ell_{coil}/N)^{\frac{1}{2}} \tag{3-7.16}$$

The wire AWG is obtained from the wire diameter over insulation from Eq. (3-2.1), and a new shape factor calculated from the AWG using Eq. (3-7.7). The new shape factor replaces the old shape factor and the calculations run again. The iteration is terminated when the new and old shape factors agree within .001.

The coil physical dimensions and number of turns have now been

determined, and other electrical characteristics can be calculated.

$$L = \frac{0.4\pi \cdot N^2 \cdot A_{iron} \cdot k_3 \times 10^{-8}}{\frac{\ell_{iron}}{\mu_{iron}} + \ell_{air} \frac{A_{iron}}{A_{air}}} \tag{3-7.17}$$

$$R = (R/\ell)(\text{mean turn})(N) \tag{3-7.18}$$

where R/ℓ is obtained from:

$$R/\ell, (\text{ohms/inch}) = k_4 \cdot e^{k_5 \cdot AWG} \tag{3-7.19}$$

hence,

$$R = N \cdot \pi \cdot (D_p + w) \cdot k_4 \cdot e^{k_5 \cdot AWG} \tag{3-7.20}$$

For the coil temperature at 60°C, the constants are:

$$\pi \cdot k_4 = 2.9185212367 \times 10^{-5}$$

$$k_5 = 0.2317635483$$

If the coil excitation is a constant voltage, then the coil current will have to be recalculated due to the downward rounding of the wire size to the nearest integral value:

$$I = \frac{E}{R} \tag{3-7.21}$$

The power dissipated in the coil is:

$$P = I^2 R \tag{3-7.22}$$

If constant voltage excitation is used, the peak flux density (B_{max}) and initial plunger attractive force will be slightly larger than the initial values again due to the downward rounding of the wire AWG. The larger wire will have lower resistance causing higher coil current and a higher NI product. Equations (3-7.15) and (3-7.1) are rearranged and used to find B_{iron} and F.

$$B_{iron} = \frac{0.4\pi\ NI}{\left\{ \frac{\ell_{iron}}{\mu_{iron}} + \ell_{air} \frac{A_{iron}}{A_{air}} \right\} k_3} \tag{3-7.23}$$

$$F = \frac{B_{max}{}^2 \cdot A_{air}}{k_1} = \frac{(B_{iron} \cdot A_{iron})^2}{k_1 \cdot A_{air}} \qquad (3\text{-}7.24)$$

In addition to the program card, a data card is necessary to load the registers with these constants:

μ_o default:	500 ⟶ R_5
B_{max} default:	15000 ⟶ R_8
Initial shape factor:	0.5 ⟶ R_E
$(\pi/4)(a'/a)$:	0.8418900745 ⟶ S_o
$2(b' - b)$:	$-1.216909380 \times 10^{-2}$ ⟶ S_1
a:	$3.130387015 \times 10^{-1}$ ⟶ S_2
b:	$-1.097333787 \times 10^{-1}$ ⟶ S_3
$\pi \cdot k_4$:	$2.985212367 \times 10^{-5}$ ⟶ S_4
k_5:	$2.317635483 \times 10^{-1}$ ⟶ S_5
k_3:	2.54 ⟶ S_6
$k_2 = \frac{4}{\pi} \times 10^6$:	1.273239545×10^6 ⟶ S_7
k_1:	1.73×10^6 ⟶ S_8
M default:	1000 ⟶ S_9

If the user wants to work in centimeter units instead of inch units, then a different set of constants can be loaded. All constants are the same except for the following:

a:	$7.951183018 \times 10^{-1}$ ⟶ S_2
$\pi \cdot k_4$:	$1.175280459 \times 10^{-5}$ ⟶ S_4
k_3:	1.0 ⟶ S_6
k_2:	5.012754114×10^5 ⟶ S_7
k_1:	1.11613×10^7 ⟶ S_8

The inverse current density, M, is now in hybrid units. The circular-mils/A must be multiplied by 2.54 before entry, and the current density, Δ, is in A/cm^2. The plunger attractive force is still in pounds. If this force is desired in kilograms, change k_1 as follows:

k_1: 2.46064×10^7 ⟶ S_8

The HP-67 user may wish the program to stop at data output points rather than executing a 5 second "print" halt. To cause the program to

stop at the data output points, change the "print" statements to "R/S" statements at the following line numbers: 047, 084, 131, 144, 160, 176, 180, 185, and 194.

User Instructions

CYLINDRICAL SOLENOID DESIGN				
B_{max} in air gap ↑ air gap (I)	ℓ_{iron} ↑ A_{iron} ↑ μ (I)	+ Volts or − Amps (I)	load M, cir mils / A (I/O)	calculate coil design and electrical parameters (O)
Force, lbs (I)	calculate pole diameter (O)	winding width (I)		

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card and both sides of data card			
2	Load force required (in pounds) at maximum air gap (plunger all the way out)	F	[A]	
3	Load maximum flux in the iron (gauss) and	B_{max}	[ENT]	
	the air gap in inches	ℓ_{air}	[f] [A]	
4	Optional, load magnetic circuit parameters:			
	a) load magnetic path length	ℓ_{iron}, in	[ENT]	
	b) load magnetic path minimum area	A_{iron}, in^2	[ENT]	
	c) load relative permeability	μ	[f] [B]	
	If this step is not executed, the program will use $A_{iron} = A_{air}$ and $\ell_{iron} = 0$			
5	Calculate pole diameter		[B]	pole diameter in inches
	To change the pole diameter, change B_{max}, a larger B_{max} will result in a smaller pole. B_{max} is material dependent, and generally should not exceed 15000 gauss.			
6	Load winding thickness	w, in	[C]	
7	Load excitation voltage or current			
	a) excitation voltage	E, V	[f] [C]	
	b) excitation current (note neg value)	−I, A	[f] [C]	
8	Load a value for M, the inverse coil current density in circular-mils per ampere. If no value is loaded, a default value of 1000 will be used. Execution of this step without numeric entry causes currently stored value to be printed and displayed.	M	[f] [D]	M Δ

User Instructions

CYLINDRICAL SOLENOID DESIGN
CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
9	Calculate coil design and electrical parameters		[E]	N AWG ℓ_{coil} L, h R, Ω P, watts B_{iron}, G B_{max}, G F, pounds

Example 3-7.1

Figure 3-7.2 shows a plunger-type, iron-clad cylindrical solenoid. Design the solenoid to have a 1 inch travel and exert an initial pull of 500 pounds when connected to a 55 volt dc source. The initial flux density in the iron shall be 7000 gauss, and the coil inverse current density shall be 700 circular-mils/A. Assume all the reluctance to be in the air gap.

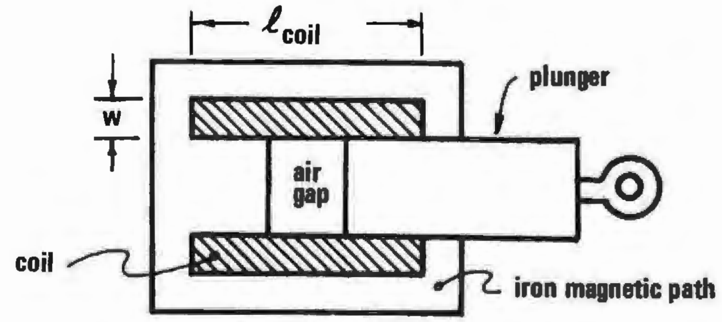


Figure 3-7.2 Plunger-type, iron-clad cylindrical solenoid.

500.00	GSBA	load initial force required in pounds (F)
7000.00	ENT↑	load maximum B field in gauss (B_{max})
1.00	GSBa	load ℓ_{air}
	GSBB	calculate plunger diameter required (D_p)
4.74	***	plunger diameter in inches
3.00	GSBC	load winding width in inches (w)
55.00	GSBc	load excitation voltage in volts (E)
700.00	GSBD	load inverse current density, M, in cir-mils/A
700.00	***	M
1815.51	***	Δ, A/in^2
	GSBE	calculate coil design and electrical parameters
1563.00	***	N, the number of turns
12.00	***	AWG of wire with heavy or class 2 insulation
3.57	***	coil length in inches (ℓ_{coil})
1.41	***	coil inductance in henries (L)
5.96	***	coil resistance in ohms (R)
512.43	***	coil power dissipation in watts (P)
7296.69	***	actual maximum flux density in the iron
7296.69	***	B_{max}, the flux density in the air gap
543.28	***	F, the plunger attractive force actually achieved

Example 3-7.2

A small solenoid is needed which has 0.050 inch travel, exerts an initial pull of 5 pounds, and is used intermittently with a 0.10 duty cycle. The coil excitation current is 3 A, and an initial flux density of 6000 gauss is to be used. Because of the intermittent duty cycle, an M of 100 cir-mils/A is used. The magnetic path is 1.5 inches long, has a cross-sectional area of 0.4 $inch^2$, and has a relative permeability of 500. Investigate the solenoid design with and without consideration for the magnetic path reluctance. A much more thorough analysis can be done with Program 3-8.

5.000	GSBA	load initial force required in pounds (F)
6000.000	ENT↑	load maximum flux density in gauss (B_{max})
.050	GSBa	load initial air gap in inches (ℓ_{air})
	GSBE	calculate plunger diameter in inches (D_p)
0.553	***	D_p
.250	GSBC	load winding width in inches (w)
-3.000	GSBc	load excitation current in A (-I)
100.000	GSBD	load inverse current density in cir-mils/A (M)
100.000	***	M
12732.395	***	Δ, $A/inch^2$
	GSBE	calculate coil design etc. without considering iron path
202.000	***	the number of turns (N)
25.000	***	AWG of coil wire with heavy insulation
0.308	***	coil length in inches (ℓ_{coil})
0.006	***	coil inductance in henries (L)
1.550	***	coil resistance in ohms (R)
14.311	***	coil power dissipation in watts (P)
5996.237	***	maximum flux density in the iron, gauss
5996.237	***	B_{max}, maximum flux density in the air gap, gauss
4.994	***	actual initial force, F, in pounds

Rerun program with magnetic (iron) path considered

1.500	ENT↑	load magnetic path length in inches
.400	ENT↑	load magnetic path area in $inches^2$
500.000	GSBb	load relative magnetic permeability
	GSBE	calculate coil design and electrical parameters
209.000	***	N
25.000	***	AWG
0.319	***	ℓ_{coil}
0.006	***	L
1.645	***	R
14.807	***	P
3597.060	***	B in iron area defined
5988.202	***	B in air gap and in iron pole pieces
4.980	***	F

Derivation of Equations Used. The number of coil turns can be calculated from the applied voltage, the desired inverse current density, and the coil inner diameter and thickness. Conveniently, copper has a resistance of 1 ohm per circular mil per inch of length at 60°C; therefore, with a uniform coil temperature of 60°C, the wire resistance is:

$$R = \frac{\ell_w}{M} \qquad (3\text{-}7.24)$$

where ℓ_w is the winding wire length in the coil in inches, and m is the wire cross-sectional area in circular mils. If M is defined as the inverse current density in circular-mils/A, then the cross-section of a wire carrying a current I is:

$$m = M \cdot I \qquad (3\text{-}7.25)$$

Since

$$R = \frac{E}{I}, \text{(Ohm's law)} \qquad (3\text{-}7.26)$$

then

$$\frac{E}{I} = \frac{\ell_w}{M \cdot I} \qquad (3\text{-}7.27)$$

Rearranging Eq. (3-7.27) and cancelling I yields:

$$E = \frac{\ell_w}{M} \qquad (3\text{-}7.28)$$

The winding wire length can be found by multiplying the mean turn length by the number of turns:

$$\ell_w = N \cdot \pi \cdot (D_p + w) \qquad (3\text{-}7.29)$$

where Fig. 3-7.1 defines the coil dimensions D_p and w. Substituting Eq. (3-7.29) into Eq. (3-7.28) and solving for N yields:

$$N = \frac{E \cdot M}{\pi(D_p + w)} \qquad (3\text{-}7.30)$$

The best reference on the subject known to the author is rather old [47] since it was first published in 1924.

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD FORCE REQUIRED
002	STO7	
003	RTN	
004	*LBLa	LOAD B_{max} ↑ ℓ_{air}
005	STO5	
006	R↓	
007	STO8	
008	GTO5	goto CF3 and return subr
009	*LBLB	CALCULATE POLE DIAMETER, D_p
010	RCL7	
011	RCL8	
012	X²	
013	÷	$A_{air} = \frac{F \cdot k_1}{B_{air}^2}$
014	P⇄S	
015	RCL8	
016	P⇄S	
017	x	
018	STO6	store air gap area
019	F1?	minimum magnetic area equals
020	STO3	airgap area if flag 1 is set
021	4	
022	x	
023	Pi	$D_p = \sqrt{\frac{4 \cdot A_{air}}{\pi}}$
024	÷	
025	√X	
026	STOD	store pole diameter
027	GTO4	goto prt, spc, & CF3 subr
028	*LBLb	LOAD ℓ_{iron} ↑ A_{iron} ↑ μ
029	CF1	indicate magnetic path used
030	STO4	
031	R↓	
032	STO3	store data
033	R↓	
034	STO2	
035	RTN	
036	*LBLC	LOAD WINDING WIDTH, w
037	STO0	
038	RTN	
039	*LBLc	LOAD COIL EXCITATION,
040	STO9	+E, or -I
041	GTO5	goto CF3 and return subr
042	*LBLD	I/O OF CIRCULAR-MILS/AMP, M
043	P⇄S	interchange registers
044	F3?	store input if present
045	STO9	
046	RCL9	recall and print M
047	PRTX	
048	RCL7	calculate and print Δ:
049	X⇄Y	$\Delta = \frac{M}{k_2}$
050	÷	
051	P⇄S	
052	GTO4	goto prt, spc, & CF3 subr
053	*LBLE	CALCULATE MAIN OUTPUT
054	RCL8	
055	RCL6	
056	x	
057	RCL3	
058	÷	calculate and store NI
059	GSB9	using Eq. (3-7.15)
060	÷	
061	STOA	
062	RCL9	
063	X<0?	test for current excitation
064	GTO0	
065	P⇄S	
066	RCL9	voltage excitation:
067	P⇄S	calculate the number of
068	x	turns using Eq. (3-7.4):
069	RCLD	
070	RCL0	
071	+	$N = \frac{E \cdot M}{\pi(D_p + w)}$
072	Pi	
073	x	
074	÷	
075	GTO1	
076	*LBL0	current excitation, calculate
077	RCLA	the number of turns using
078	X⇄Y	Eq. (3-7.5):
079	÷	$N = (NI)/I$
080	CHS	
081	*LBL1	calculate, store and print
082	INT	the integral number of turns
083	STOI	
084	PRTX	
085	*LBL7	iteration loop start
086	RCLA	calculate and store coil
087	RCLE	length using Eq. (3-7.11):
088	÷	
089	RCL0	
090	÷	
091	P⇄S	
092	RCL9	$\ell_{coil} = \frac{NI \cdot M}{sf \cdot k_2 \cdot w}$
093	x	
094	RCL7	
095	P⇄S	
096	÷	
097	STO1	
098	RCLI	
099	1/X	calculate wire diameter over
100	RCL0	insulation using Eq. (3-7.16)
101	x	
102	RCL1	$d = \sqrt{\frac{w \cdot \ell_{coil}}{N}}$
103	x	
104	√X	
105	P⇄S	
106	RCL2	calculate and store wire AWG
107	÷	using Eq. (3-2.1)
108	LN	
109	RCL3	$AWG = \frac{1}{b} \ln \left\{ \frac{\text{wire diameter}}{a} \right\}$
110	÷	

REGISTERS

0	1	2	3	4	5	6	7	8	9
w	ℓ_{coil}	ℓ_{iron}	A_{iron}	μ	ℓ_{air}	A_{air}	F	B_{max}	+volts or -amps
S0 $\frac{\pi}{4}\left(\frac{a'}{a}\right)^2$	S1 $2(b'-b)$	S2 a	S3 b	S4 $\pi \cdot k_4$	S5 k_5	S6 $k_3 = 2.54$	S7 $k_2 = \frac{4 \times 10^6}{\pi}$	S8 $k_1 = 1.73 \times 10^6$	S9 M

A	B	C	D	E	I
NI, I	AWG	R	Dp	sf	N

Program Listing II

NOTE FLAG SET STATUS

Keys	Comments
STOE	
RCL1 x e^x RCL0 P⇄S x	calculate shape factor using Eq. (3-7.7): $sf = \frac{\pi}{4}\left(\frac{a'}{a}\right)^2 e^{AWG\cdot 2(b'-b)}$
RCLE X⇄Y STOE	recall old sf and store new sf
− ABS EEX CHS 3 X≤Y? GTO7	test for loop exit
RCLB INT STOB PRTX	print & store integral AWG
RCL1 PRTX	recall and print the number of turns
SF2	indicate k_3 on top
GSB9 RCLI X^2 x RCL3 x EEX 8 ÷ PRTX	calculate and print inductance using Eq. (3-7.17) $L = \frac{0.4\pi\cdot N^2\cdot A_{iron}\cdot k_3\cdot 10^{-8}}{\frac{\ell_{iron}}{\mu_{iron}} + \frac{\ell_{air}\cdot A_{iron}}{A_{air}}}$
RCLB P⇄S RCL5 x e^x RCL4 P⇄S x RCLD RCL0 + x RCL1 x STOC PRTX	calculate and print resistance using Eq. (3-7.20) $R = N\pi(D_p + w)\,k_4\,e^{k_5\cdot AWG}$
RCL9 X<0? GTO0	test for current excitation
RCLC ÷	calculate current using Ohm's law
*LBL0 ABS STOA X^2 RCLC x PRTX	calculate and print coil power dissapation using Eq. (3-7.21): $P = I^2R$
GSB9 RCLA x RCLI x PRTX	calculate and print new B_{iron} using Eq. (3-7.22): $B_{iron} = \frac{0.4\pi NI/k_3}{\frac{\ell_{iron}}{\mu_{iron}} + \frac{\ell_{air}\cdot A_{iron}}{A_{air}}}$
RCL3 x RCL6 ÷ PRTX	calculate and print: $B_{max} = \frac{B_{iron}\cdot A_{iron}}{A_{air}}$
X^2 RCL6 x P⇄S RCL8 P⇄S ÷	calculate and print new F using Eq. (3-7.23): $F = \frac{B_{max}^2\cdot A_{air}}{k_1}$
*LBL4 PRTX SPC	print, spc, CF3 subroutine
*LBL5 CF3 RTN	CF3 and return subroutine
*LBL9 RCL2 RCL4 ÷ RCL5 RCL3 x RCL6 ÷ + 1/X	magnetics subroutine to calculate: $\frac{0.4\pi}{\frac{\ell_{iron}}{\mu_{iron}} + \frac{\ell_{air}\cdot A_{iron}}{A_{air}}}\cdot\frac{k_3(F2=1)}{k_3(F2=0)}$
7 2 D+R	generates 0.4π in 3 steps
x P⇄S RCL6 P⇄S F2? 1/X ÷ RTN	

NOTE FLAG SET STATUS

LABELS					FLAGS
A load F	B calc Dp	C load w	D load M	E calc coil & elect params	0
a B_{max} ↑ air gap	b ℓ_{iron} ↑ A_{iron} ↑ μ	c load +E, −I	d	e	1 magnetic path data not loaded
0 Local label	1 Local label	2	3	4 output routine	2 conversion order
5 CF3, RTN	6	7 sf iteration routine	8	9 subroutine	3 data entry

SET STATUS			
FLAGS	ON	OFF	TRIG / DISP
0			DEG ■ / FIX ■
1	■		GRAD / SCI
2		■	RAD / ENG
3		■	n 2

PROGRAM 3-8 CYLINDRICAL SOLENOID ANALYSIS.

Program Description and Equations Used

This program analyzes a cylindrical coil solenoid, or other magnetic circuits having many parts of varying reluctance. The information required to run the program is as follows:

1) The air gap in inches (ℓ_{air}),
2) The number of turns in the coil (N),
3) The AWG of the coil wire,
4) The length of the coil in inches (ℓ_{coil}),
5) The coil inner diameter in inches (ID_{coil}),
6) The plunger outer diameter in inches (OD_p),
7) The plunger inner diameter in inches if the plunger is hollow (ID_p),
8) The length, area, and permeability of each different magnetic section (ℓ_{iron}, A_{iron}, μ),

8a) If the magnetic section is a cylindrical shell with axial flux flow, the height (h), the ID which may be zero, the OD, and the permeability (μ), can be entered, and the reluctance and cross-sectional area will be returned and automatically loaded into the program,

8b) If the magnetic section consists of a disc (or washer) with radial flux flow, the thickness (t), the ID, the OD, and the permeability can be entered, and the reluctance and minimum cross-sectional area will be returned and automatically loaded into the program, and

9) The coil excitation in either volts or amperes (E or −I).

The program will then calculate the following parameters:

1) Reluctance and area of each different magnetic section ($\mathcal{R}$ & A_{iron}),
2) Coil inductance and resistance (R and L),
3) Coil circular-mils/A, A/in^2, and power dissipation M, Δ, & P),
4) The flux density in the air gap, and in the magnetic section with the smallest cross-sectional area (B_{air}, B_{iron}), and
5) The plunger attractive force in pounds (F).

This program uses the Ohm's law of magnetics as given by Eqs. (3-7.12) and (3-7.13), which combined yield:

$$0.4\pi NI = \phi \cdot \sum_i \frac{\ell_i}{\mu_i A_i} \quad (3\text{-}8.1)$$

As magnetic path data is entered, the program keeps a running sum of the reluctances, $\frac{\ell_i}{\mu_i A_i}$, and also stores the smallest magnetic area. The iron part will saturate first where the area is the smallest, and the flux density (B) the highest. The total flux can be found from Eq. (3-8.1):

$$\phi = \frac{0.4\mu N I k_3}{\sum\limits_{\substack{\text{iron}\\\text{parts}}} \frac{\ell_i}{\mu_i A_i} + \frac{\ell_{air}}{A_{air}}} \quad (3\text{-}8.2)$$

where

$$A_{air} = \frac{\pi}{4}\left(OD_p^{\,2} - ID_p^{\,2}\right) \quad (3\text{-}8.3)$$

$$k_3 = 2.54$$

The plunger attractive force is found in terms of the flux:

$$F = \frac{\phi^2}{k_1 \cdot k_3^4 \cdot A_{air}} \quad (3\text{-}8.4)$$

where the air gap area is in inches^2 and the constant k_1 is:

$$k_1 = 1.73 \times 10^6$$

The inductance of the N turn coil wound on the magnetic circuit is:

$$L = \frac{N^2 k_3}{10^8}\left\{\frac{0.4\pi}{\sum\limits_{\substack{\text{iron}\\\text{parts}}} \frac{\ell}{\mu A} + \frac{\ell_{air}}{A_{air}}}\right\} \quad (3\text{-}8.5)$$

This expression is basically derived in Eqs. (3-1.1) through (3-1.10).

The coil width (w) can be expressed in terms of the coil length (ℓ_{coil}), the number of turns (N), and the wire AWG. The wire is assumed to occupy a box as shown in Fig. 3-6.2.

$$\text{coil area} = w \cdot \ell_{coil} = N \cdot (\text{wire diameter})^2 \quad (3\text{-}8.6)$$

Substituting the exponential relationship between AWG and wire diameter given by Eq. (3-5.10) yields:

$$w = \frac{N}{\ell_{coil}}\left(a \cdot e^{b \cdot AWG}\right)^2 \quad (3\text{-}8.7)$$

The coil resistance can now be calculated using Eq. (3-7.20):

$$R = N \cdot \pi\left(ID_{coil} + w\right)\left(k_4\, e^{k_5 \cdot AWG}\right)$$

The coil power dissipation is:

$$P = I^2 R \quad (3\text{-}8.8)$$

If voltage excitation is used, the coil current is calculated using Ohm's law, then the power dissipation is calculated.

The coil circular mils per A is given by:

$$M = \underbrace{10^6 \cdot \left(a' \cdot e^{b' \cdot AWG}\right)^2}_{\substack{\text{wire area in}\\\text{circular mils}}} \Big/ I \quad (3\text{-}8.9)$$

The coil current density in A/in^2 is given by Eq. (3-8.10), i.e.:

$$\Delta = \frac{k_2}{M} \quad (3\text{-}8.10)$$

Two commonly encountered part shapes in the magnetic path are the cylindrical shell as shown in Fig. 3-8.1 and the disc or washer as shown in Fig. 3-8.2. Two subroutines are provided to calculate the reluctance and minimum cross-sectional area of these two shapes.

Subroutine 1, thin cylindrical shell with permeability μ.

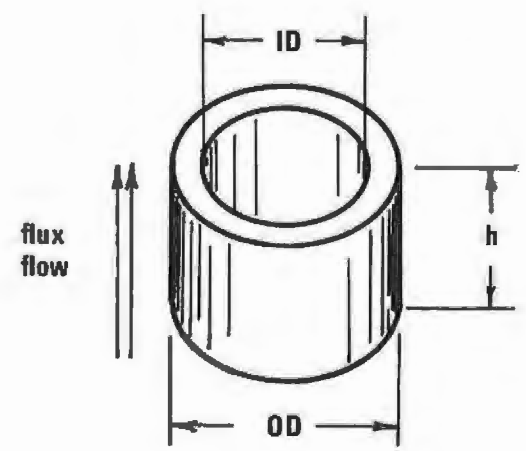


Figure 3-8.1 Thin cylindrical shell.

The cross-sectional area is given by Eq. (3-8.3) and the reluctance is:

$$\mathcal{R} = \frac{h}{\mu A}$$

This subroutine output becomes the input for the program coding under label B, and the reluctance is calculated under label B. The subroutine output is stored in the stack in the same format as data entered from the keyboard for arbitrary magnetic section, i.e.:

stack register	contents
t	not used
z	h
y	cross-sectional area
x	permeability

Subroutine 2, disc or washer with radial flux flow.

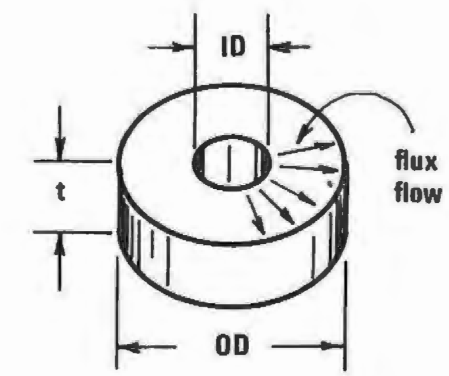


Figure 3-8.2 Disc or washer with radial flux flow.

The disc is composed of an infinite number of annular shells each with infinitesimal thickness dr. The cross-sectional area of each annulus is $2\pi rt$. In this instance, the summation of Eq. (3-8.1) is expressed as an integral:

$$\mathcal{R} = \sum\frac{\ell}{\mu A} = \frac{1}{\mu t}\int_{r_1=\frac{ID}{2}}^{r_2=\frac{OD}{2}} \frac{dr}{2\pi r} = \frac{\ell n(OD/ID)}{2\pi t\mu} \quad (3\text{-}8.11)$$

The disc has the smallest cross-sectional area at the inner diameter, hence:

$$A = A' = \pi \cdot ID \cdot t \quad (3\text{-}8.12)$$

This subroutine output becomes the input for the program coding under label B. The data format used with label B is the equivalent length of a constant cross-section magnetic path, the path area, and the path permeability. The equivalent length having the above reluctance and cross-sectional area A' is:

$$\ell = \mu A' \cdot \mathcal{R} = \left(\frac{\pi \cdot ID \cdot t \cdot \mu}{2\pi \cdot t \cdot \mu}\right) \cdot \ell n \frac{OD}{ID} = \frac{ID}{2} \ell n \frac{OD}{ID} \tag{3-8.13}$$

Subroutine 2 output is transferred to the program coding under label B using the stack in the same way that subroutine 1 operates.

In addition to the program card, a data card is required to load the registers with the program constants. All registers contain zero except for the following:

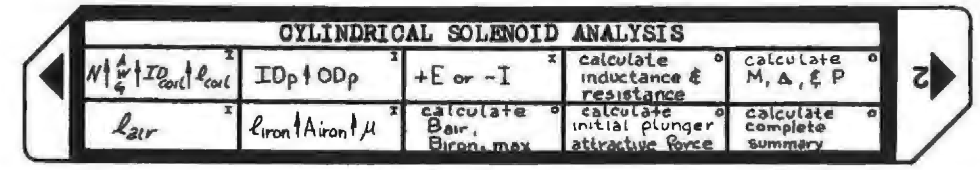
a' for AWG	$3.241013109 \times 10^{-1} \longrightarrow S_0$
b' for AWG	$-1.158179256 \times 10^{-1} \longrightarrow S_1$
a for AWG	$3.130387015 \times 10^{-1} \longrightarrow S_2$
b for AWG	$-1.097333787 \times 10^{-1} \longrightarrow S_3$
$\pi \cdot k_4$ for resistance	$2.985212367 \times 10^{-5} \longrightarrow S_4$
k_5 for resistance	$2.317635483 \times 10^{-1} \longrightarrow S_5$
k_3, cm $\rightarrow$ inch	$2.54 \longrightarrow S_6$
k_2, $4/\pi \times 10^6$	$1.273239545 \times 10^6 \longrightarrow S_7$
k_1	$1.73 \times 10^6 \longrightarrow S_8$

If metric units are preferred, i.e., linear dimensions in cm, force in kg, current density in A/cm^2 and inverse current density in hybrid units (circular mil-milli-centimeter/A), change the following constants.

a' for AWG	$8.232173297 \times 10^{-1} \longrightarrow S_0$
a for AWG	$7.951183108 \times 10^{-1} \longrightarrow S_2$
$\pi \cdot k_4$ for resistance	$1.175280459 \times 10^{-5} \longrightarrow S_4$
k_3 cm $\rightarrow$ cm	$1.0 \longrightarrow S_6$
k_2, $4/(2.54\pi) \times 10^6$	$5.012754114 \times 10^5 \longrightarrow S_7$
k_1	$2.4606 \times 10^7 \longrightarrow S_8$

HP-67 users may want the program to stop instead of executing a "print" statement. This can be accomplished by changing the "print" statements to "R/S" statements at the following line numbers: 102, 105, 124, and 130. To continue program execution after a stop, key a "R/S" command from the keyboard.

3-8 **User Instructions**



STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card and both sides of data card			
2	Load air gap length in inches	ℓ_{air}	[A]	
3	Load plunger ID and OD in inches. The ID can be zero if the plunger is solid	ID_P OD_P	[ENT↑] [f] [B]	
4	Load coil parameters:			
	number of wire turns in coil	N	[ENT↑]	
	wire AWG	AWG	[ENT↑]	
	coil ID in inches	ID_{coil}	[ENT↑]	
	coil length in inches	ℓ_{coil}	[f] [A]	
5	Load coil excitation			
	voltage excitation in volts	E	[f] [C]	
	current excitation in A (note minus)	-I	[f] [C]	
6	Optional step, the main source of reluctance in the magnetic path is the air gap. For added accuracy, the length, area, and permeability of each magnetic section may be entered:			
	effective magnetic path length in inches	ℓ_{iron}	[ENT↑]	
	effective magnetic path area in inches^2	A_{iron}	[ENT↑]	
	magnetic permeability of path	μ	[B]	$\mathcal{R}$ A
	If the magnetic section is either a cylindrical shell or a disc, then a subroutine can be used to calculate and enter the above parameters from the section dimensions.			
	For cylindrical shells with axial flux flow:			
	load shell height in inches	h	[ENT↑]	
	load shell ID in inches (may be zero)	ID	[ENT↑]	
	load shell OD in inches	OD	[ENT↑]	
	load shell permeability	μ	[GSB] [1]	$\mathcal{R}$ A

3-8 User Instructions

CYLINDRICAL SOLENOID ANALYSIS

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
6	continued			
	For discs with radial flux flow:			
	load disc thickness in inches	t	ENT ↑	
	load disc ID in inches	ID	ENT ↑	
	load disc OD in inches	OD	ENT ↑	
	load permeability of material	μ	GSB 2	$\mathcal{R}$
				A
	Repeat step 6 for each separate magnetic section in the magnetic circuit.			
7	To calculate the flux density in the air gap and in the smallest iron cross-sectional area (the smallest area has the highest flux dens)		C	B_{air}, G
				B_{iron}, G
	If step 6 is omitted, $\mathcal{R}_{iron} = 0$, $A_{iron} = A_{air}$ is assumed, hence, $B_{iron} = B_{air}$			
8	To calculate the initial plunger attractive force in pounds		D	F
9	To calculate the electrical inductance and resistance at 60°C of the coil		f D	L, h
				R, ohms
10	To calculate the coil M, Δ, and power dissipation		f E	M, $\frac{\text{cir-mils}}{\text{A}}$
				Δ, A/in^2
				P, watts
11	To calculate all the information contained in steps 8, 9, 10, and 11		E	L, h
				R, ohms
				M, $\frac{\text{cir-mils}}{\text{A}}$
				Δ, A/in^2
				P, watts
				B_{air}, G
				B_{iron}, G
				F, lbs
12	To run a new case, goto step 1 and start over			

Example 3-8.1

The cylindrical solenoid shown in cross-section by Fig. 3-8.3 has the following characteristics:

1) The coil is 150 turns of #24 AWG HF wire,
2) 0.5 A excitation current flows through the coil, and
3) The magnetic materials are 1010 mild carbon steel.

For the analysis, neglect the force required to compress the return spring.

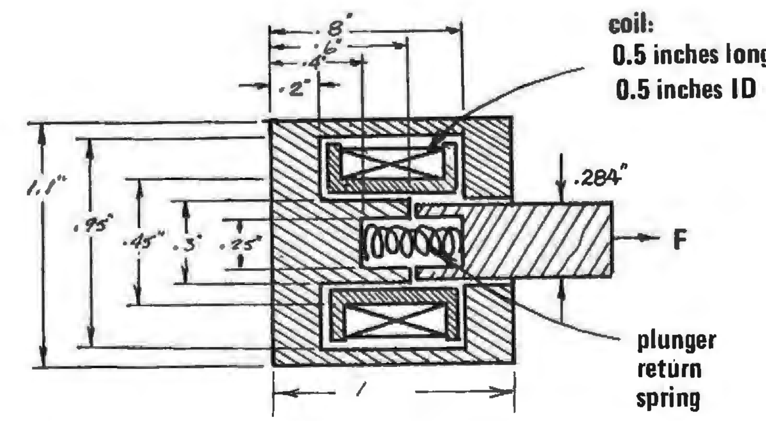


Figure 3-8.3 Cylindrical solenoid construction.

Analyze the solenoid and determine its electrical and magnetic characteristics. Also analyze the solenoid for the same characteristics if the coil is excited by 0.6 Vdc.

The analysis is begun by breaking down the solenoid into its component geometric shapes as shown by Fig. 3-8.4.

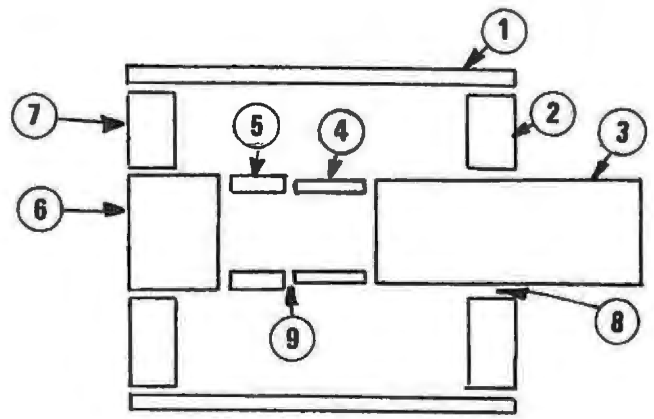


Figure 3-8.4 Component geometric shapes of solenoid.

The component geometric shapes of the solenoid are as follows:

1) Cylindrical shell, 1.0" long, 0.95" ID, 1.1" OD, and μ= 1000,
2) Disc, 0.2" thick, 0.3" ID, 0.95" OD, and μ = 1000,
3) Solid cylinder, 0.2" long (active magnetic part), 0.0" ID, 0.284" OD, and μ = 1000,
4) Cylindrical shell, 0.2" long, 0.25" ID, 0.284" OD, and μ = 1000,
5) Cylindrical shell, 0.2" long, 0.25" ID, 0.3" OD, and μ = 1000,
6) Solid cylinder, 0.4" long, 0.0" ID, 0.3" OD, and μ = 1000,
7) Disc, 0.2" thick, 0.3" ID, 0.95" OD, and μ = 1000,
8) Disc (air gap), 0.2" thick, 0.284" ID, 0.3" OD, and μ = 1,
9) Operating air gap, 0.005" thick, 0.25" ID, 0.284" OD, & μ = 1.

The air gap data is loaded, and the complete summary calculated, then the magnetic path component parts are loaded and the summary run again to show the difference that the magnetic circuit reluctance makes on the electrical and magnetic characteristics. This sequence is repeated with the coil excitation at 0.6 Vdc.

HP-97 PRINTOUT FOR EXAMPLE 3-8.1

```
     .005 GSBA load ℓair

     150. ENT↑ load N
      24. ENT↑ load AWG
       .5 ENT↑ load ID
       .5 GSBa load ℓcoil

      .25 ENT↑ load IDp
     .284 GSBb load ODp

      -.5 GSBc load -I

               calc all *
          GSBE parameters
2.048-03   *** L, h
759.9-03   *** R, ohms

809.2+00   *** M, cir-mils/A
1.574+03   *** Δ, A/in²
190.0-03   *** P, watts

7.421+03   *** max Biron
7.421+03   *** Bair, G

453.9-03   *** F, pounds

 * Magnetic reluctance
   is assumed zero
   since flag 0 is set.
   Flag 0 is cleared
   under label B.

load magnetic path data

 cylindrical shell
       1. ENT↑ length
      .95 ENT↑ ID
      1.1 ENT↑ OD
    1000. GSB1 μ

4.141-03   *** ℛ
241.5-03   *** area

          disc
       .2 ENT↑ thickness
       .3 ENT↑ ID
      .95 ENT↑ OD
    1000. GSB2 μ

917.3-06   *** ℛ
188.5-03   *** area
```

```
solid cylinder
       .2 ENT↑ length
       0. ENT↑ ID
     .284 ENT↑ OD
    1000. GSB1 u

3.157-03   *** ℛ
63.35-03   *** area

 cylindrical shell
       .2 ENT↑ length
      .25 ENT↑ ID
     .284 ENT↑ OD
    1000. GSB1 μ

14.03-03   *** ℛ
14.26-03   *** area

 cylindrical shell
       .2 ENT↑ length
      .25 ENT↑ ID
       .3 ENT↑ OD
    1000. GSB1 μ

9.260-03   *** ℛ
21.60-03   *** area

   solid cylinder
       .4 ENT↑ length
       0. ENT↑ ID
       .3 ENT↑ OD
    1000. GSB1 μ

5.659-03   *** ℛ
70.69-03   *** area

          disc
       .2 ENT↑ thickness
       .3 ENT↑ ID
      .95 ENT↑ OD
    1000. GSB2 μ

917.3-06   *** ℛ
188.5-03   *** min area

          disc
       .2 ENT↑ thickness
     .284 ENT↑ ID
       .3 ENT↑ OD
       1. GSB2 μ

43.62-03   *** ℛ
178.4-03   *** min area
```

```
          GSBE calc all params
1.661-03   *** L, h
759.9-03   *** R, ohms

809.2+00   *** M, cir-mils/A
1.574+03   *** Δ, A/in²
190.0-03   *** P, watts

6.019+03   *** max Biron
6.019+03   *** Bair, G

298.6-03   *** F, pounds


 Look at voltage excita-
 tion.  Set flag 0 so
 magnetic reluctance is
 ignored and calculate
 electrical & magnetic
 parameters.
           SF0
       .6 GSBc load E

          GSBE calc params
2.048-03   *** L, h
759.9-03   *** R, ohms

512.4+00   *** M, cir-mils/A
2.485+03   *** Δ, A/in²
473.7-03   *** P, watts

11.72+03   *** max Biron
11.72+03   *** Bair, G

1.132+00   *** F, pounds

 Clear flag 0 to use
 magnetic reluctance.
           CF0
          GSBE calc params
1.661-03   *** L, h
759.9-03   *** R, ohms

512.4+00   *** M, cir-mils/A
2.485+03   *** Δ, A/in²
473.7-03   *** P, watts

9.505+03   *** max Biron
9.505+03   *** Bair, G

744.6-03   *** F, pounds
```

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD AIR GAP IN INCHES
002	STO2	store entry
003	GTO0	goto space and return
004	*LBLa	LOAD N ↑ AWG ↑ ID_{coil} ↑ ℓ_{coil}
005	STO1	store coil length
006	R↓	recover and store ID_{coil}
007	STOD	
008	R↓	recover and store AWG
009	STOB	
010	R↓	recover and store N
011	STOE	
012	GTO0	goto space and return
013	*LBL2	SUBROUTINE FOR DISC
014	R↓	
015	X⇄Y	
016	STOI	
017	÷	
018	LN	$\ell_{effective} = \frac{ID}{2} \ln \frac{OD}{ID}$
019	RCLI	
020	x	
021	2	
022	÷	
023	X⇄Y	
024	RCLI	
025	x	$A_{min} = \pi \cdot ID \cdot t$
026	Pi	
027	x	
028	R↑	recover μ
029	*LBLB	LOAD ℓ_{iron} ↑ A_{iron} ↑ μ
030	X⇄Y	store μ
031	STOI	
032	F0?	store A_{iron} on first
033	STO4	execution of this routine
034	x	
035	÷	$\mathcal{R}_i = \frac{\ell_{iron_i}}{\mu_i \cdot A_{iron_i}}$
036	SPC	
037	PRTX	
038	ST+5	add $\mathcal{R}_i$ to Σ
039	RCL4	test to see if present area
040	RCLI	is smaller than minimum
041	X≤Y?	stored area, if so, store
042	STO4	present area
043	CF0	indicate magnetic params
044	GSB8	print area and space
045	GTO0	goto space and return
046	*LBLb	LOAD ID_p ↑ OD_p
047	GSB4	calculate and store
048	STO3	annular area
049	GTO0	goto space and return

NOTE:

The "print" statements at line numbers 037, 102, 105, 124, and 130 may be changed to "R/S" statements if desired.

Step	Key	Comment
050	*LBLC	CALCULATE B_{air} and B_{iron}
051	GSB6	calculate R, I, and NI
052	GSB9	calc $(0.4\pi/k_3)/(\Sigma\mathcal{R}_i + \ell_{air}/A_{air})$
053	RCL4	use A_{air} if magnetic params
054	F0?	not entered, otherwise use
055	RCL3	min Airon
056	1/X	take reciprocal area
057	RCLI	calculate and print ϕ
058	RCLA	using Eq. (3-7.2)
059	x	
060	STO6	
061	x	calculate and print:
062	P⇄S	
063	RCL6	
064	P⇄S	
065	X²	$B_{iron,max} = \frac{\phi}{\min A_{iron} \cdot k_3^2}$
066	STOI	
067	÷	
068	PRTX	
069	RCL6	calculate and print:
070	RCL3	
071	÷	$B_{max} = \frac{\phi}{A_{air} \cdot k_3^2}$
072	RCLI	
073	÷	
074	GTO8	
075	*LBLE	PRINT COMPLETE SUMMARY
076	GSBd	
077	GSBe	
078	GSBC	
079	*LBLD	CALCULATE AND PRINT F
080	GSB6	
081	SF2	
082	GSB9	
083	RCLA	
084	x	
085	X²	
086	RCL3	$F = \frac{\phi^2}{k_1 \cdot k_3^4 \cdot A_{air}}$
087	÷	
088	P⇄S	
089	RCL8	
090	P⇄S	
091	÷	
092	GTO8	
093	*LBLd	CALCULATE AND PRINT L & R
094	GSB6	
095	GSB9	
096	RCLE	
097	X²	
098	x	$L = \frac{N^2 k_3}{10^8} \cdot \frac{0.4\pi}{\sum_{iron}\mathcal{R} + \frac{\ell_{air}}{A_{air}}}$
099	EEX	
100	8	
101	÷	
102	PRTX	
103	RCLC	recall and print resistance

REGISTERS

0 w	1 ℓ_{coil}	2 ℓ_{air}	3 A_{air}	4 min A_{iron}	5 $\sum \frac{\ell}{\mu A}$	6 ϕ	7 scratch	8 I	9 E
S0 a′	S1 b′	S2 a	S3 b	S4 πk_4	S5 k_5	S6 k_3	S7 k_2	S8 k_1	S9

A NI	B AWG	C R	D coil OD	E N	I scratchpad

Program Listing II

NOTE FLAG SET STATUS

Step	Key	Comment
104	*LBL8	print and space subroutine
105	PRTX	
106	GTO0	
107	*LBLc	LOAD COIL EXCITATION
108	STO9	
109	*LBL0	space and return subroutine
110	SPC	
111	RTN	
112	*LBLe	CALCULATE AND PRINT M, ,P
113	GSB6	
114	P⇄S	
115	RCL0	
116	RCL1	
117	GSB3	
118	X²	
119	RCL8	$M = 10^6 \left(a' e^{b' \cdot AWG}\right)^2$
120	÷	
121	EEX	
122	6	
123	x	
124	PRTX	
125	1/X	
126	P⇄S	
127	RCL7	$\Delta = \frac{k_2}{M}$
128	P⇄S	
129	x	
130	PRTX	
131	RCL8	
132	X²	
133	RCLC	$P = I^2 R$
134	x	
135	GTO8	
136	*LBL1	CYLINDRICAL SHELL SUBR
137	STOI	
138	R↓	
139	GSB4	
140	RCLI	
141	GTOB	
142	*LBL3	subroutine to calculate:
143	P⇄S	
144	RCLB	
145	x	$R_y \cdot e^{R_x \cdot AWG}$
146	e^x	
147	x	
148	RTN	
149	*LBL4	subroutine to calculate:
150	X²	
151	X⇄Y	
152	X²	
153	−	
154	Pi	$Area = \frac{\pi}{4}(OD^2 - ID^2)$
155	x	
156	4	
157	÷	
158	RTN	

Step	Key	Comment
159	*LBL6	subr to calc R, I, and NI
160	P⇄S	
161	RCL2	
162	RCL3	
163	GSB3	
164	X²	
165	RCLE	$W = \frac{N}{\ell_{coil}}\left(a \cdot e^{b \cdot AWG}\right)^2$
166	x	
167	RCL1	
168	÷	
169	STO0	
170	RCLD	
171	+	
172	RCLE	
173	x	
174	P⇄S	
175	RCL4	$R = N\pi(ID_{coil} + W)k_4 e^{k_5 \cdot AWG}$
176	RCL5	
177	GSB3	
178	x	
179	STOC	
180	RCL9	
181	X<0?	test for current excitation
182	GTO0	
183	÷	$I = E/R$
184	1/X	
185	*LBL0	jump destination
186	ABS	store $\lvert I \rvert$
187	STO8	
188	RCLE	calculate and store NI
189	x	
190	STOA	
191	RTN	
192	*LBL9	magnetics subroutine
193	7	0.4π
194	2	
195	D→R	
196	RCL2	
197	RCL3	
198	÷	
199	RCL5	
200	F0?	
201	CLX	
202	+	$\frac{0.4\pi}{\sum_{iron}\mathcal{R}_i + \frac{\ell_{air}}{A_{air}}} \cdot \frac{k_3 \ (F2=0)}{k_5 \ (F2=1)}$
203	÷	
204	P⇄S	
205	RCL6	
206	P⇄S	
207	F2?	
208	1/X	
209	x	
210	STOI	
211	RTN	return to main program

NOTE FLAG SET STATUS

LABELS

A ℓ_{air}	B ℓ_{iron} ↑ A_{iron} ↑ μ	C calculate B_{air}, max B_{iron}	D calc F	E complete summary
a N ↑ AWG ↑ ID_{coil} ↑ ℓ_{coil}	b ID_p ↑ OD_p	c +E or −I	d calc L, R	e calc M, Δ, & P
0 local label	1 cylindrical section entry	2 disc section entry	3 wire size subroutine	4 circular section area
5	6 R, I & NI subroutine	7	8 print & space subroutine	9 magnetics calc subr

FLAGS

0 magnetic parameters entered
1
2 subroutine 9 control
3

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0	■		users choice	
1			DEG	FIX
2		■	GRAD	SCI
3			RAD	ENG ■
				n 3

PROGRAM 3-9 MAGNETIC RELUCTANCE OF TAPERED CYLINDRICAL SECTIONS.

Program Description and Equations Used

This program calculates the magnetic reluctance of tapered cylindrical sections with axial flux flow as shown by Fig. 3-9.1. The magnetic reluctance is analogous to electrical resistance, and is used in the Ohm's law of magnetics as given by Eq. (3-8.1).

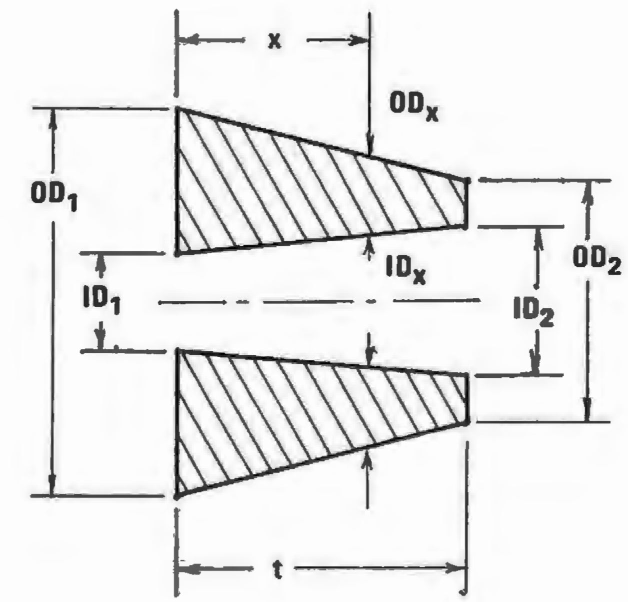


Figure 3-9.1 Tapered cylindrical section and dimensions.

Consider the section to be composed of an infinite number of washers each of infinitesimal thickness dx, then the reluctance of a washer is:

$$d\mathcal{R} = dx/(\mu \cdot A_x) \qquad (3\text{-}9.1)$$

where

$$A_x = (\pi/4)(OD_x^2 - ID_x^2) \qquad (3\text{-}9.2)$$

The inner and outer diameters at location x can be found by linearly interpolating between the known end diameters:

$$ID_x = ID_1 + (1/t)(ID_2 - ID_1)\cdot x \qquad (3\text{-}9.3)$$

$$OD_x = OD_1 + (1/t)(OD_2 - OD_1)\cdot x \qquad (3\text{-}9.4)$$

Substituting Eqs. (3-9.3) and (3-9.4) into Eq. (3-9.2) and collecting like powers of x results in a quadratic:

$$A_x = (\pi/4)(a + bx + cx^2) \qquad (3\text{-}9.5)$$

where

$$a = OD_1{}^2 - ID_1{}^2$$

$$b = (2/t)\{OD_1(OD_2 - OD_1) - ID_1(ID_2 - ID_1)\}$$

$$c = (1/t^2)\{(OD_2 - OD_1)^2 - (ID_2 - ID_1)^2\}$$

hence,

$$\mathcal{R} = \frac{4}{\mu\pi}\int_0^t \frac{dx}{a + bx + cx^2} \qquad (3\text{-}9.6)$$

The result of this integration can have any one of three forms; let

$$q = b^2 - 4ac \qquad (3\text{-}9.7)$$

and

$$r = (2cx + b)/\sqrt{|q|} \qquad (3\text{-}9.8)$$

then if $q>0$ and $|r|<1$, the solution is:

$$\mathcal{R} = -\frac{8}{\mu\pi\sqrt{|q|}}\tanh^{-1} r \qquad (3\text{-}9.9)$$

if $q>0$ and $|r| \geq 1$, the solution is:

$$\mathcal{R} = \frac{4}{\mu\pi\sqrt{|q|}}\ \ell n\left(\frac{r-1}{r+1}\right) \qquad (3\text{-}9.10)$$

if $q < 0$, the solution for all r is:

$$\mathcal{R} = \frac{8}{\mu\pi\sqrt{|q|}}\tan^{-1} r \qquad (3\text{-}9.11)$$

3-9 # User Instructions

MAGNETIC RELUCTANCE OF TAPERED CYLINDRICAL SECTIONS				
load $ID_1 \uparrow ID_2$	load $OD_1 \uparrow OD_2$	load section length, t	load permeability, μ	print ? / calculate $\mathcal{R}$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of the magnetic card			
2	select print/ no-print option		f E	0 (no prt)
			f E	1 (print)
			f E	0 (no prt)
				⋮
3	Load inner diameters	ID_1*	ENT↑	
		ID_2*	A	
4	Load outer diameters	OD_1*	ENT↑	
		OD_2*	B	
5	Load section length	t*	C	
6	Load magnetic permeability of material		D	
7	Calculate reluctance		E	$\mathcal{R}$**

Notes

* Any units of the users choosing may be used as long as the same unit is used throughout. If the reluctance is going to be loaded into Program 3-7, then inch units should be used.

** The units of reluctance are in inverse dimension units, i.e., $inches^{-1}$, cm^{-1}, ft^{-1}, etc.

Example 3-9.1

Given the conical section shown in Fig. 3-9.2, calculate the reluctance in inch units.

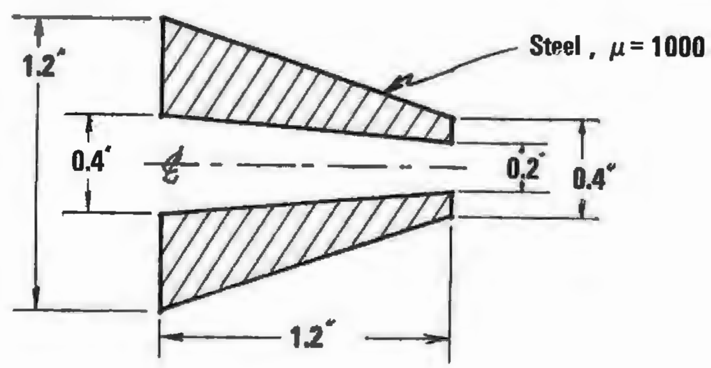


Figure 3-9.2 Tapered conical section.

```
      .2 ENT↑   ID1
      .4 GSBA   ID2

      .4 ENT↑   OD1
      .2 GSBB   OD2

      .3 GSBC   t

   1000. GSBD   μ

         GSBE   calculate reluctance
3.872-03  ***   ℛ, in^-1
```

3-9 **Program Listing I**

```
001  *LBLA   LOAD ID1 ↑ ID2
002   STO1
003     R↓   store entries
004   STO0
005   GTO0   goto space and return subr
006  *LBLB   LOAD OD1 ↑, OD2
007   STO3
008     R↓   store entries
009   STO2
010   GTO0   goto space and return subr
011  *LBLC   LOAD SECTION LENGTH
012   STO4
013   GTO0
014  *LBLD   LOAD PERMEABILITY
015   STO5
016  *LBL0   space and return subroutine
017    SPC
019    RTN
```

```
019  *LBLE   CALCULATE RELUCTANCE
020   RCL3   calculate and store:
021   RCL2
022      -     (OD2 - OD1)^2
023     X²
024   STO8
025   LSTX   calculate and retain in stk:
026   RCL2     OD1 (OD2 - OD1)
027      x
028   RCL1   calculate w/ register arith:
029   RCL0
030      -
031   ENT↑     (OD2 - OD1)^2 - (ID2 - ID1)^2
032     X²
033   ST-8
034     R↓   calculate and store b:
035   RCL0
036      x
037      -
038   ENT↑     2/t {OD1(OD2 - OD1) - ID1(ID2 - ID1)}
039      +
040   RCL4
041      ÷
042   STO7
043   RCL4   finish c calculation
044     X²
045   ST÷8
046   RCL7   calculate and store q:
047     X²
048   RCL2
049     X²
050   RCL0
051     X²
052      -     q = b^2 - 4ac
053   RCL8
054      x
055      4
056      x
057      -
058   STOB
059    ABS   calculate and store:
060     √X     √|q|
061   STOA
062   RCL4   calculate and store:
063   GSB0
064   STO9     ∫_0^t dx / (a + bx + cx^2)
065    CLX
066   GSB0
067   ST-9
```

REGISTERS

0	1	2	3	4	5	6	7	8	9
ID_1	ID_2	OD_1	OD_2	t	μ	scratch	b	c	$\int^t$
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
$\sqrt{\lvert q \rvert}$	q	2cx + b		$\mathcal{R}$	

Program Listing II

Step	Key	Comment
068	RCL9	calculate and store reluctance
069	4	
070	x	$\mathcal{R} = \frac{4}{\mu\pi}\int_0^t \frac{dx}{a + bx + cx^2}$
071	RCL5	
072	Pi	
073	x	
074	÷	
075	STOE	
076	F1?	print reluctance and space if flag 1 is set
077	PRTX	
078	F1?	
079	SPC	
080	RTN	
081	*LBL0	integral evaluation subroutine
082	ENT↑	
083	+	
084	RCL8	calculate and store r:
085	x	
086	RCL7	
087	+	$r = \frac{2cx + b}{\sqrt{\lvert q \rvert}}$
088	STOC	
089	RCLA	
090	÷	
091	STO6	
092	ABS	set flag 0 if the magnitude of r is greater than 1
093	EEX	
094	SF0	
095	X>Y?	
096	CF0	
097	RCLB	jump if q is less than 0
098	X<0?	
099	GTO0	
100	F0?	jump if flag 0 is set
101	GTO2	
102	EEX	calculate $\tanh^{-1} r$
103	RCL6	
104	+	
105	EEX	
106	RCL6	$\tanh^{-1} r = \frac{1}{2} \ln \frac{1+r}{1-r}$
107	-	
108	÷	
109	√X	
110	LN	
111	CHS	change sign per Eq. (3-8.9)
112	GTO1	jump
113	*LBL2	logarithmic solution
114	RCLC	
115	RCLA	
116	-	
117	RCLC	
118	RCLA	
119	+	
120	÷	
121	LN	
122	RCLA	
123	÷	
124	RTN	return to main program
125	*LBL0	trigonometric solution
126	RCL6	calculate $\tan^{-1} r$
127	TAN⁻¹	
128	*LBL1	common portion of hyperbolic and trigonometric solutions
129	ENT↑	
130	+	
131	RCLA	
132	÷	
133	RTN	return to main program
134	*LBLe	PRINT OR R/S TOGGLE
135	F1?	jump if flag 1 is set
136	GTO3	
137	SF1	set flag 1
138	EEX	place 1 in display
139	RTN	return control to keyboard
140	*LBL3	clear flag 1 and place a zero in the display
141	CF1	
142	CLX	
143	RTN	return control to keyboard

Flag 1 should be set (cleared) before magnetic card recording depending upon the user's desire for the program to normally be in the print (R/S) mode after the card read.

LABELS					FLAGS
A load $ID_1 \uparrow ID_2$	B load $OD_1 \uparrow OD_2$	C load t	D load permeability	E calculate reluctance	0 r > 1
a	b	c	d	e print / R/S toggle	1 print
0 local label	1 subroutine destination	2 subroutine destination	3 print / R/S destination	4	2
5	6	7	8	9	3

SET STATUS		
FLAGS (ON OFF)	TRIG	DISP
0 OFF ■	DEG	FIX
1	GRAD	SCI
2	RAD ■	ENG ■
3		n 3

Part 4

HIGH FREQUENCY CIRCUIT DESIGN

PROGRAM 4-1 BILATERAL TRANSISTOR AMPLIFIER DESIGN USING S PARAMETERS.

Program Description and Equations Used

When s_{12}, the reverse transmission coefficient, cannot be reduced to near zero using unilateral design methods,* or the unilateral figure of merit is not sufficiently near zero, the bilateral design method must be used. Since s_{12} is related to the capacitive reactance of the transistor base-collector capacity, and this reactance becomes smaller as frequency increases, the bilateral design requirement generally occurs when the amplifier is to be used at UHF frequencies and above.

The bilateral stability factor, K, is computed using Eq. (4-1.1). For the amplifier to be unconditionally stable, K must be greater than one, and the magnitudes of s_{11} and s_{22} must be smaller than one. Since s_{11} and s_{22} are reflection coefficients, this last requirement implies that the input and output impedances are positive. Unconditional stability means the amplifier will not oscillate for any choice of input and output terminations.

$$K = \frac{1 + |\Delta|^2 - |s_{11}| - |s_{22}|}{2\,|s_{21} \cdot s_{12}|} \qquad (4\text{-}1.1)$$

$$\Delta = s_{11} \cdot s_{22} - s_{21} \cdot s_{12} \qquad (4\text{-}1.2)$$

When K is less than one, the amplifier will oscillate with certain source and load impedances, hence, these impedances must be carefully selected. The HP EE pac Program 18 will calculate the stability circles to aid in the termination impedance selection.

The scattering parameters are:

s_{11} is the input reflection coefficient,
s_{12} is the reverse transmission coefficient,
s_{21} is the forward transmission coefficient, and
s_{22} is the output reflection coefficient.

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* See the HP EE pac Program 16 for unilateral design methods.

Scattering parameters are obtained from reflection coefficient measurements applied to a two port network with both ports loaded with a reference impedance, Z_o, which is typically 50 ohms resistive. The reflection coefficient is defined by Eq. (1-1.2). For a more comprehensive discussion of s parameters, see Froehner [24], HP application note 95 [32], or Carson [15].

If the proposed amplifier is unconditionally stable, then the maximum gain can be calculated using Eq. (4-1.3)

$$G_{max} = \left|\frac{s_{21}}{s_{12}}\right| \cdot (K \pm \sqrt{K^2 - 1}) \qquad (4\text{-}1.3)$$

The negative sign is used when B_1 is positive and vice-versa:

$$B_1 = 1 + |s_{11}|^2 - |s_{22}|^2 - |\Delta|^2 \qquad (4\text{-}1.4)$$

The source and load reflection coefficients necessary to provide G_{max} are given by Eqs. (4-1.5) and (4-1.6). These loads present a conjugate match to the transistor.

$$\rho_{MS} = C_1^* \cdot \frac{B_1 \pm \sqrt{B_1^2 - 4|C_1|^2}}{2|C_1|^2} \qquad (4\text{-}1.5)$$

$$\rho_{ML} = C_2^* \cdot \frac{B_2 \pm \sqrt{B_2^2 - 4|C_2|^2}}{2|C_1|^2} \qquad (4\text{-}1.6)$$

$$B_2 = 1 + |s_{22}|^2 - |s_{11}|^2 - |\Delta|^2 \qquad (4\text{-}1.7)$$

$$C_1 = s_{11} - \Delta \cdot s_{22}^* \qquad (4\text{-}1.8)$$

$$C_2 = s_{22} - \Delta \cdot s_{11}^* \qquad (4\text{-}1.9)$$

The minus sign in Eqs. (4-1.5) and (4-1.6) is used when B_1 is positive and vice-versa. The asterisk (*) means the complex conjugate, i.e., the sign of the imaginary part is reversed, or the sign of the angle is reversed for rectangular or polar formats respectively.

Equations (4-1.5) and (4-1.6) are used to calculate reflection coefficients. The corresponding impedances can be obtained if Eq. (1-1.2) is rearranged to provide Z_L in terms of Z_s:

$$Z_L = Z_o \frac{1 + \rho}{1 - \rho} \qquad (4\text{-}1.10)$$

This routine is contained under label E of the program.

4-1 **User Instructions**

BILATERAL TRANSISTOR AMPLIFIER DESIGN USING S PARAMETERS				
		calculate ρ_{Source}, max		print, R/S toggle
load $\Theta_{ij} \uparrow s_{ij} \uparrow ij$	calculate K, G_{max}	calculate ρ_{Load}, max		$\angle\rho \uparrow |\rho| \uparrow Z_o \rightarrow Im, Re\ Z$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Select print or R/S option		f E	0 (R/S)
			f E	1 (print)
			f E	0 (R/S)
				⋮
3	Load elements of s parameter matrix for ij = 11, 12, 21, 22 (any order)			
	a) load angle of s_{ij} in degrees	$\Theta_{ij}°$	ENT↑	
	b) load magnitude of s_{ij}	$|s_{ij}|$	ENT↑	
	c) load subscript	ij	A	
4	Calculate stability factor and maximum gain		B	K
				G_{max}, dB
5	Calculate angle and magnitude of load reflection coefficient to obtain G_{max}		C	$\angle\rho_{mL}$
				$|\rho_{mL}|$
	Calculate real and imaginary parts of load impedance	Z_o	E	Re Z_L
				Im Z_L
6	Calculate angle and magnitude of source reflection coefficient to obtain G_{max}		f C	$\angle\rho_{ms}$
				$|\rho_{ms}|$
	Calculate real and imaginary parts of source impedance	Z_o	E	Re Z_s
				Im Z_s
7	Calculate real and imaginary parts of impedances corresponding to a reflection coefficient and Z_o	$\angle\rho$	ENT↑	
		$|\rho|$	ENT↑	
		Z_o	E	Re Z
				Im Z

<u>Example 4-1.1</u>

Given a 2N3570 transistor operating at I_c = 4 mA and V_{ce} = 10 V and having the following s parameters at 750 MHz,

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 0.277 \measuredangle -59° & 0.078 \measuredangle 93° \\ 1.920 \measuredangle 60° & 0.848 \measuredangle -31° \end{bmatrix}$$

calculate the stability factor, the maximum power gain in dB, the source reflection coefficient and impedance to obtain G_{max}, and the load reflection coefficient and impedance to obtain G_{max}.

PROGRAM INPUT		PROGRAM OUTPUT	
-59.000 ENT↑	θ_{11}, angle in degrees	GSBB	calculate K & G_{max}
.277 ENT↑	s_{11}, magnitude	1.035+00 ***	K > 1, uncond stable
11. GSBA	ij	12.51+00 ***	G_{max}, dB
93.000 ENT↑	θ_{12}		
.078 ENT↑	s_{12}	GSBc	calculate ρ_{MS}
12. GSBA	ij	135.4+00 ***	$\measuredangle\rho_{MS}$, degrees
64.000 ENT↑	θ_{21}	729.6-03 ***	$\lvert\rho_{MS}\rvert$
1.920 ENT↑	s_{21}	50. GSBE	calculate Z_s
21. GSBA	ij	9.063+00 ***	Re Z_s, ohms
		19.90+00 ***	Im Z_s, ohms
-31.000 ENT↑	θ_{22}		
.848 ENT↑	s_{22}		
22. GSBA	ij	GSBC	calculate ρ_{ML}
		33.85+00 ***	$\measuredangle\rho_{ML}$, degrees
		551.1-03 ***	$\lvert\rho_{ML}\rvert$
		50. GSBE	calculate Z_L
		14.59+00 ***	Re Z_L, ohms
		163.1+00 ***	Im Z_L, ohms

4-1

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD $\theta_{ij} \uparrow s_{ij} \uparrow ij$
002	ENT↑	
003	+	calculate storage register index
004	2	
005	1	
006	-	
007	STOI	
008	R↓	recover and store s_{ij}
009	STOi	
010	ISZI	increment register index
011	R↓	recover and store θ_{ij}
012	STOi	
013	GTO3	goto space and return
014	*LBLB	CALCULATE K, G_{max}
015	RCL2	calculate and store $s_{11} \cdot s_{22}$
016	RCL1	
017	RCLD	
018	RCLE	
019	GSB9	
020	STO6	$\lvert s_{11} \cdot s_{22}\rvert$
021	R↓	
022	STO7	$\measuredangle s_{11} s_{22}$
023	RCL4	calculate and store:
024	RCL3	
025	RCLB	
026	RCLC	
027	GSB9	$\Delta = s_{11} \cdot s_{22} - s_{12} \cdot s_{21}$
028	CHS	
029	RCL7	
030	RCL6	
031	GSB8	
032	STO6	$\lvert\Delta\rvert$
033	R↓	
034	STO7	$\measuredangle\Delta$
035	RCLD	finish K calculation
036	RCL1	
037	GSB7	$B_1 = \lvert s_{11}\rvert^2 - \lvert s_{22}\rvert^2 - \lvert\Delta\rvert^2 + 1$
038	RCL6	
039	X²	
040	EEX	
041	+	
042	RCL1	
043	X²	
044	-	
045	RCLD	
046	X²	
047	-	$1 + \lvert\Delta\rvert^2 - \lvert s_{11}\rvert^2 - \lvert s_{22}\rvert^2$
048	RCL3	
049	RCLB	
050	×	
051	ABS	
052	ENT↑	
053	+	
054	÷	
055	STO9	$K = \dfrac{1 + \lvert\Delta\rvert^2 - \lvert s_{11}\rvert^2 - \lvert s_{22}\rvert^2}{2\lvert s_{21} \cdot s_{12}\rvert}$
056	GSB5	goto print routine
057	X²	calculate G_{max}:
058	LSTX	
059	X⇄Y	
060	EEX	
061	-	
062	√X	
063	RCL5	$G_{max} = \left\lvert\dfrac{S_{21}}{S_{12}}\right\rvert \cdot (K \pm \sqrt{K^2 - 1})$
064	×	
065	-	
066	RCLB	– used when B_1 is +
067	×	
068	RCL3	
069	÷	
070	ABS	convert G_{max} to dB power gain
071	LOG	
072	EEX	
073	1	
074	×	
075	GTO0	goto print and space subr
076	*LBLc	CALCULATE $\rho_{source,\,max}$
077	RCL7	calculate $-\Delta \cdot s_{22}$
078	RCL6	
079	RCLD	
080	RCLE	
081	CHS	
082	GSB9	
083	CHS	
084	RCL2	recall s_{11}
085	RCL1	
086	GSB2	calc and print $\measuredangle\rho_{source,\,max}$
087	RCLD	$\lvert s_{22}\rvert$
088	RCL1	$\lvert s_{11}\rvert$
089	GSB7	calc $\lvert s_{11}\rvert^2 - \lvert s_{22}\rvert^2 - \lvert\Delta\rvert^2 + 1 = B_1$
090	GTO1	jump
091	*LBLC	CALCULATE $\rho_{load,\,max}$
092	RCL7	calculate $-\Delta \cdot s_{11}$
093	RCL6	
094	RCL1	
095	RCL2	
096	CHS	
097	GSB9	
098	CHS	
099	RCLE	recall s_{22}
100	RCLD	
101	GSB2	calc and print $\measuredangle\rho_{load,\,max}$
102	RCL1	$\lvert s_{11}\rvert$
103	RCLD	$\lvert s_{22}\rvert$
104	GSB7	calc $\lvert s_{22}\rvert^2 - \lvert s_{11}\rvert^2 - \lvert\Delta\rvert^2 - 1 = B_2$
105	*LBL1	calculate refl coef mag
106	RCLA	$-\measuredangle B$
107	RCL8	$\lvert B\rvert$
108	RCL0	$\lvert C\rvert$
109	ENT↑	
110	+	

REGISTERS

0 $\lvert C\rvert$	1 $\lvert s_{11}\rvert$	2 $\measuredangle s_{11}$	3 $\lvert s_{12}\rvert$	4 $\measuredangle s_{12}$	5 sign(B_1)	6 scratch, $\lvert\Delta\rvert$	7 scratch, $\measuredangle\Delta$	8 scratch, B_2 or B_1	9 K
S0 Re ρ	S1 Im ρ	S2 Z_o, Z_s	S3 $\measuredangle Z_s$	S4	S5	S6	S7	S8	S9

A $\measuredangle C^*$, $-\measuredangle B$	B $\lvert s_{21}\rvert$	C $\measuredangle s_{21}$	D $\lvert s_{22}\rvert$	E $\measuredangle s_{22}$	I index

Program Listing II

Step	Key	Comments
111	÷	
112	ENT↑	$\lvert\rho\rvert = \frac{B}{2C} - (\text{sign } B)\sqrt{\left(\frac{B}{2C}\right)^2 - 1}$
113	X²	
114	EEX	
115	-	
116	√X	
117	RCL5	
118	×	
119	-	
120	*LBL0	**print and space subroutine**
121	GSB5	
122	*LBL3	**space subroutine**
123	F0?	**space if flag 0 is set**
124	SPC	
125	GTO6	**goto R/S lock**
126	*LBLE	CONVERT $\angle\rho\uparrow\lvert\rho\rvert\uparrow Z_o \to \text{Im}, \text{Re } Z$
127	P⇄S	
128	STO2	
129	R↓	
130	→R	
131	STO0	Re ρ
132	EEX	
133	+	
134	X⇄Y	
135	STO1	Im ρ
136	X⇄Y	
137	→P	$1+\rho$
138	ST×2	$Z_o \cdot (1+\rho)$
139	X⇄Y	
140	STO3	$\angle(1+\rho)$
141	RCL1	
142	CHS	$-\text{Im } \rho$
143	EEX	
144	RCL0	Re ρ
145	-	
146	→P	$1-\rho$
147	ST÷2	$\lvert Z_o\rvert \cdot \lvert(1+\rho)/(1-\rho)\rvert = \lvert Z\rvert$
148	X⇄Y	
149	ST-3	$\angle(1+\rho) - \angle(1-\rho) = \angle Z$
150	RCL3	$\angle Z$
151	RCL2	$\lvert Z\rvert$
152	P⇄S	
153	→R	**convert to rectangular fmt**
154	GSB5	**print Re Z**
155	X⇄Y	**recover Im Z**
156	GTO0	**print Im Z and space**
157	*LBL7	**subroutine to calculate:**
158	X²	
159	X⇄Y	$X^2 - Y^2 - \lvert\Delta\rvert^2 + 1 = B$
160	X²	
161	-	sign(B) → R5
162	EEX	
163	+	
164	RCL6	
165	X²	

Step	Key	Comments
166	-	
167	STO8	
168	X=0?	
169	EEX	
170	ABS	
171	LSTX	
172	÷	
173	STO5	
174	RTN	
175	*LBL8	**complex add subroutine**
176	→R	
177	R↓	
178	R↓	
179	→R	
180	X⇄Y	
181	R↓	
182	+	
183	R↓	
184	+	
185	R↑	
186	→P	
187	RTN	
188	*LBL9	**complex multiply subroutine**
189	R↓	
190	×	
191	R↓	
192	+	
193	R↑	
194	RTN	
195	*LBL2	**subroutine to finish C**
196	GSB8	**calculation, store results**
197	STO0	**and print angle of**
198	X⇄Y	**reflection coefficient**
199	CHS	
200	STOA	
201	*LBL5	**print subroutine**
202	F0?	
203	PRTX	**print and return if flag 0**
204	F0?	**is set, otherwise stop**
205	RTN	
206	R/S	
207	RTN	
208	*LBL6	**R/S lock**
209	R/S	**prevents inadvertent use**
210	GTO6	**of program fcns w/ R/S**
211	*LBLe	**PRINT, R/S TOGGLE**
212	CF0	**clear flag 0 to indicate**
213	CLX	**R/S mode and place a zero**
214	RTN	**in the display**
215	*LBLe	
216	SF0	**set flag 0 to indicate**
217	EEX	**print and continue mode**
218	RTN	**and place a one in display**

NOTE FLAG SET STATUS

LABELS					FLAGS
A $\theta_{ij}\uparrow s_{ij}\uparrow ij$	B → K, G_{max}	C → ρ_{mL}	D	E $\rho\uparrow Z_o \to Z$	0 print
a	b	c → ρ_{ms}	d	e	1
0 prt, spc, rtn	1 local label w/ C, fC	2 subroutine w/ C, fC	3 space, rtn	4	2
5 print, R/S subroutine	6 R/S lock	7 subroutine	8 complex add	9 complex multiply	3

SET STATUS		
FLAGS	TRIG	DISP
ON OFF 0 ■ 1 2 3	DEG ■ GRAD RAD	FIX SCI ENG ■ n 3

PROGRAM 4-2 UHF OSCILLATOR DESIGN USING S PARAMETERS.

Program Description and Equations Used

At UHF frequencies, the interelement capacities of a UHF transistor can function as the feedback elements to allow the device to oscillate when connected to an external tuned circuit (usually a ¼-wave transmission line section). The emitter circuit is generally left unbypassed while the base circuit is bypassed with a capacitor to provide an ac ground. The collector-emitter capacity provides the necessary feedback to allow the collector to exhibit negative output impedance and oscillate with the external tuned circuit.

The program starts with the common base s parameters, reverses the port ordering so the collector is the input, and calculates the reflection coefficient of the "input." If the magnitude of the reflection coefficient is greater than one, the real part of the input impedance will be negative. The routine under label E provides the conversion from reflection coefficient to impedance, while the routine under label e provides the reverse conversion.

Equation (4-2.1) calculates the input reflection coefficient when the output port is loaded with R_L as shown by Fig. 4-2.1. Equation (4-2.1) holds for any transistor configuration.

$$s_{11}' = s_{11} + \frac{s_{12}\cdot s_{21}\cdot \rho_L}{1 - s_{22}\cdot \rho_L} \quad (4\text{-}2.1)$$

where ρ_L is defined by Eq. (1-1.2) with $Z_r = R_L$.

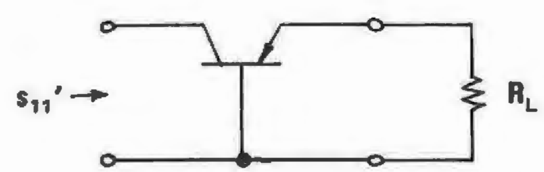

Figure 4-2.1 Common base transistor with collector as input port.

If the tuned source is connected to the collector, and the reflection coefficient of the source is denoted ρ_s, the circuit will oscillate if:

$$\rho_s \cdot s_{11}' \geq 1 \tag{4-2.2}$$

This equation is used in reverse to calculate the source reflection coefficient necessary for oscillation, i.e.:

$$\rho_s = \frac{1}{s_{11}'} \tag{4-2.3}$$

This reflection coefficient can be converted to its equivalent impedance using Eq. (4-1.10). The "Q," or quality factor, of this impedance is the ratio of the imaginary part to the real part, i.e.:

$$Q = \frac{\text{Im } Z_s}{\text{Re } Z_s} \tag{4-2.4}$$

The transistor negative input impedance can also be used to make a reflection amplifier if a circulator is used to separate the input from the output. The noise figure will be poor because of the large unbypassed emitter resistance.

For more information see the HP Journal [33], or HP application note number 95 [32].

Notes for User Instructions. Most UHF transistors are four lead devices (emitter, base, collector, and case). The case is electrically isolated from the transistor, in fact, the transistor chip is so small that it is mounted on the end of the collector lead inside the case. Because

of the fourth element, the case, the parasitic capacities from it to the other leads will introduce errors into the common-emitter to common-base s parameter conversion. See G. Bodway's article [9] on characterization of transistors by means of three port scattering parameters as one way of dealing with this problem.

If the common base s parameters are available, or can be measured, they are the highly preferred form of data input for the program. Common-base parameters notwithstanding, the common-emitter conversion can be used with the knowledge that s_{11}' will not be very accurate.

User Instructions

UHF OSCILLATOR DESIGN USING S PARAMETERS

interchange ports 1&2	fA, C, fD, E	print s matrix	calculate $1/S_{11}' = \rho_s$	Im Z ↑ Re Z ↑ Z_o → ∡ρ, |ρ|
load data θ_{ij}↑|s_{ij}|↑ij	load ∡ρ_L↑|ρ_L|	calculate s_{11}'	load ∡ρ_s↑|ρ_s| calc $s_{11}' \cdot \rho_s$	∡ρ↑|ρ|↑Z_o → Im Z, Re Z

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card or load Program 4-3 if parameter conversion reqd			
2	Load s parameters. If already in common base form, goto step 10 after executing this step.			
	a) load angle of scattering parameter	θ_{ij}	ENT↑	
	b) load magnitude of scattering parameter	|s_{ij}|	ENT↑	
	c) load subscript of scattering parameter	ij	A	
	Repeat this step for ij = 11, 12, 21, 22 in any order			
3	To convert common emitter s parameters to common base, load EE1-06A, parameter conversions: s⇄Y, G, Z, H. (see notes in step 16)			
4	Convert s parameters to Y parameters	Z_o	B	
5	Load Program 4-3 to convert common emitter Y parameters to common base Y parameters			
6	Perform CE to CB conversion		B	
7	Reload EE1-06A to convert Y parameters back to s parameters			
8	Convert Y parameters to s parameters	Z_o	f B	
9	Reload both sides of this program card (4-2)			
10	Calculate load reflection coefficient			
	a) load imaginary part of $Z_{emitter}$	Im Z_L	ENT↑	
	b) load real part of $Z_{emitter}$	Re Z_L	ENT↑	
	c) load reference impedance	Z_o	f E	∡ρ_L
				|ρ_L|

User Instructions

UHF OSCILLATOR DESIGN USING S PARAMETERS

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
11	Enter load reflection coefficient	∡ρ_L	ENT↑	
	(if step 10 is used, the reflection coefficient magnitude and angle are already in the stack--- use key "B" alone)	|ρ_L|	B	
12	Interchange port ordering 1⇄2		f A	
13	Calculate s_{11}'		C	∡s_{11}'
				|s_{11}'|
14	Calculate $\rho_s = 1/s_{11}'$		f D	∡$1/s_{11}'$
				|$1/s_{11}'$|
15	Convert ρ_s to Z_s: enter reference impedance	Z_o	E	Im Z_s
				Re Z_s
	To find the minimum resonator Q		÷	Q_{min}
16	The load reflection coefficient is not erased when Program 4-3 or EE1-06A* is used, hence, for another case, the keystrokes in steps 12, 13, 14, and 15 are contained in user definable key fB, therefore, for another case, do steps 1 through 9, then execute fB		f B	∡s_{11}
				|s_{11}|
	* In HP EE pac (supplied by HP)			∡$1/s_{11}'$
				|$1/s_{11}'$|
				Im Z_s
				Re Z_s
				Q_{min}

Example 4-2.1

A UHF oscillator using a RCA 2N5179 transistor is to operate between 300 MHz and 400 MHz. The transistor is to be operated at 1.5 mA collector current and 4 volts V_{ce} per the manufacturer's recommendations. At 300 MHz the common-emitter y parameters are:

$$\begin{bmatrix} \{(6.5 + j9.0) \times 10^{-3}\} & \{-j1.35 \times 10^{-3}\} \\ \{(32 - j32) \times 10^{-3}\} & \{(0.25 + j2.6) \times 10^{-3}\} \end{bmatrix}$$

and at 400 MHz the common-emitter y parameters are:

$$\begin{bmatrix} \{(9.2 + j10.7) \times 10^{-3}\} & \{-j1.8 \times 10^{-3}\} \\ \{(25 - j34) \times 10^{-3}\} & \{(0.3 + j4.0) \times 10^{-3}\} \end{bmatrix}$$

The proposed oscillator schematic is shown in Fig. 4-2.2, and biasing networks have been added to achieve the manufacturer's recommended bias. The 100 ohm resistor in series with the RFC lowers the Q of the resonant circuit formed by the RFC and the coax capacity so the circuit will not preferentially oscillate at that lower frequency.

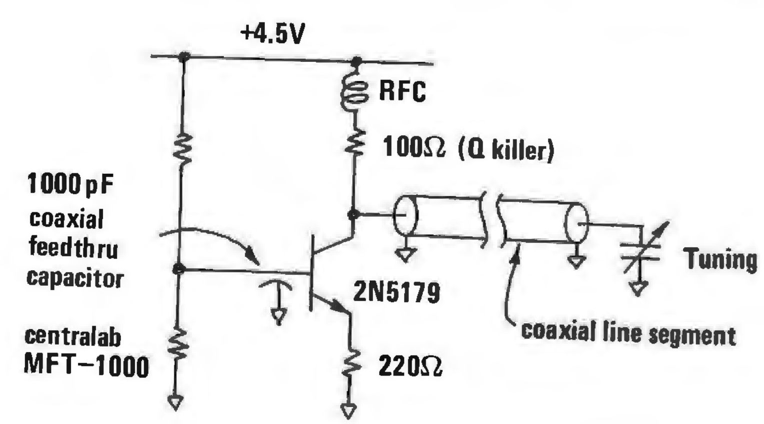

Figure 4-2.2 Oscillator schematic for Example 4-2.1.

HP-97 PRINTOUT FOR EXAMPLE 4-2.1, 300MHz CASE

```
Load Program 4-3 & load y params
        1. GSBe   select y parameters
- - - - - - - - - - - - - - - - - - - -
        9. ENT↑   Im   load yie and
      6.5    →P   Re   convert to
     1.-03    x        polar format
       11. GSBA   ij
- - - - - - - - - - - - - - - - - - - -
      -90. ENT↑   θ    load yre in
   1.35-03        mag  polar format
       12. GSBA   ij
- - - - - - - - - - - - - - - - - - - -
      -32. ENT↑   Im   load yfe and
            CHS   Re   convert to
             →P        polar format
     1.-03    x
       21. GSBA   ij
- - - - - - - - - - - - - - - - - - - -
      2.6 ENT↑    Im   load yoe and
      .25    →P   Re   convert to
     1.-03    x        polar format
       22. GSBA   ij
- - - - - - - - - - - - - - - - - - - -
           GSBE   print stored params
  54.16+00  ***   ∡y11   or  ∡yie
  11.10-03  ***   |y11|      |yie|

 -90.00+00  ***   ∡y12   or  ∡yre
  1.350-03  ***   |y12|      |yre|

 -45.00+00  ***   ∡y21   or  ∡yfe
  45.25-03  ***   |y21|      |yfe|

  84.51+00  ***   ∡y22   or  ∡yoe
  2.612-03  ***   |y22|      |yoe|
- - - - - - - - - - - - - - - - - - - -
           GSBB   CE → CB conversion
- - - - - - - - - - - - - - - - - - - -
           GSBE   print stored params
 -29.31+00  ***   ∡yib
  44.44-03  ***   |yib|

  78.69+00  ***   ∡yrb
 -1.275-03  ***   |yrb|

 -42.35+00  ***   ∡yfb
 -43.64-03  ***   |yfb|

  84.51+00  ***   ∡yob
  2.612-03  ***   |yob|
```

```
Load Program EE1-06A (EE pac)
       50. GSBb  load reference Z &
                 convert y params to
                 s parameters
----------------------------------------
load this program (Program 4-2)

           GSBc  print s parameters
 147.6+00   ***  ∡s11
 464.4-03   ***  |s11|

 93.63+00   ***  ∡s12
 41.01-03   ***  |s12|

-27.41+00   ***  ∡s21
 1.404+00   ***  |s21|

-10.62+00   ***  ∡s22
 1.024+00   ***  |s22|
- - - - - - - - - - - - - - - - - - - -
                 load RL and calculate
                 ρL using 50 ohm Zo

        0. ENT↑  Im RL
      220. ENT↑  Re RL
       50. GSBe  Zo
 0.000+00   ***  ∡ρL
 629.6-03   ***  |ρL|
- - - - - - - - - - - - - - - - - - - -
           GSBB  load ρL into program
- - - - - - - - - - - - - - - - - - - -
           GSBb  execute design
-9.018+00   ***  ∡s11'
 1.027+00   ***  |s11'|

 9.018+00   ***  ∡1/s11'
 973.4-03   ***  |1/s11'|

 616.0+00   ***   Im ZL   for Zo = 50 Ω
 105.9+00   ***   Re ZL

 5.817+00   ***  Qmin = Im ZL / Re ZL
```

A transmission line segment is designed to provide the load reactance of j616 ohms to resonate at 300 MHz. The real part of the load reactance is ignored since the Q of the resonant line will be much larger than the minimum Q required. The amplitude of the oscillation will increase until the amplifier becomes non-linear and its power gain is reduced to the point that Eq. (4-2.2) is satisfied with the equals sign.

Because of the high load reactance required, a high Z_o in the resonant line is desired. For the transmission line, use a #12 AWG wire spaced 0.25" off a ground plane as shown by Fig. 4-2.3.

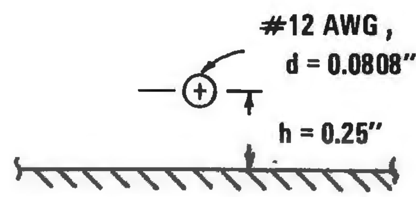

Figure 4-2.3 Air dielectric transmission line.

The characteristic impedance, Z_o, of this line is:

$$Z_o = \frac{138}{\sqrt{\varepsilon_r}} \log \frac{4h}{d} \tag{4-2.5}$$

where ε_r is the relative dielectric constant of the dielectric, and is unity for air. Using this ε_r, and the d and h shown in Fig. 4-2.3, the characteristic impedance of the line is 150.6 ohms.

If the trimmer capacitor at the far end of the line is a 1 - 10 pF piston trimmer, its reactance with 10 pF at 300 MHz is:

$$X_c = -j/(2\pi f C) = -\ j53.05 \text{ ohms} \tag{4-2.6}$$

The length of transmission line that transforms -j53.05 ohms to j616 ohms is needed. Equation (1-1.1) can be manipulated to provide the solution for line length i.e.:

$$e^{2\gamma \ell} = \frac{\rho}{\rho_t} \tag{4-2.7}$$

where ρ_t is defined by Eq. (1-2.7). Since the transmission line load impedance is purely imaginary, as is the required input impedance, and the line is essentially lossless, the expressions for the reflection coefficients are the ratios of complex conjugates, and Eq. (4-2.7) can be reduced to the following forms:

$$\ell = \frac{\lambda}{2\pi}\left\{\tan^{-1}\left(\frac{j\cdot Z_r}{Z_o}\right) - \tan^{-1}\left(\frac{j\cdot Z_s}{Z_o}\right)\right\} \tag{4-2.8}$$

where

$$\gamma = j\beta = j\frac{2\pi}{\lambda}$$

Using Eq. (4-2.8) with Z_r = -j53.05 ohms, Z_s = 616 ohms, Z_o = 150.6 ohms, and $\lambda = 3 \times 10^8/\text{freq}$ = 1 meter = 39.27 inches yields ℓ = 10.46 inches. This length is too long to be practical. If capacity is added to the transistor collector circuit, less inductance will be required from the transmission line stub, and a shorter stub can be used. If 10 pF is added from the collector to ground, the susceptance of this capacitor will be:

$$B = 2\pi f C = (2\pi)(300 \times 10^6)(10^{-11}) = 18.85 \text{ mmho}$$

This susceptance is subtracted from the susceptance required from the transmission line stub to obtain the new transmission line susceptance and hence, input reactance:

$$B_{line} = \frac{-1}{616} - 0.01885 = -\ 0.02047 \text{ mho}$$

or

$$X_{line} = \frac{-1}{B_{line}} = 48.84 \text{ ohms}$$

Using Eq. (4-2.8) with Z_s = j48.84 and the other parameters unchanged yields ℓ = 4.09 inches, which is much more practical. With this line length, the trimmer capacitor value for oscillation at 400 MHz is calculated as shown by the HP-97 printout in Fig. (4-2.5). Again, neglecting the real part of Z_L, and accommodating the susceptance of the additional 10 pF at the transistor collector, the line must present a reactance of 36.22 ohms to the collector. Using Eq. (4-2.8) and solving for Z_r given ℓ = 4.09 inches, λ = 29.53 inches, Z_s = j36.22 ohms and Z_o = 150.6 ohms yields Z_r = -j110.8 ohms. At 400 MHz, -j110.8 ohms is the reactance of a 3.6 pF capacitor, which is within the tuning range of the piston trimmer capacitor. The complete schematic of the oscillator is shown in Fig. 4-2.4, which was breadboarded and does oscillate over the 300 to 400 MHz range. This type of oscillator is often used as the local oscillator in UHF tv tuners.

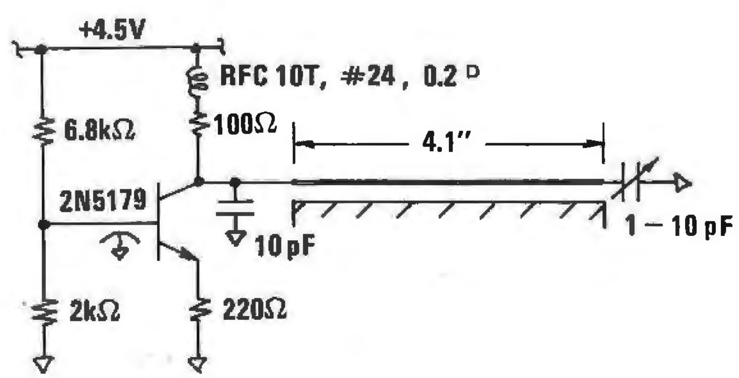

Figure 4-2.4 UHF oscillator schematic.

Load Program 4-3 and load y params		
1.	GSBe	select y parameters
10.7	ENT↑	Im y_{ie}
9.2	→P	Re y_{ie}
1.-03	x	
11.	GSBA	ij
-90.	ENT↑	$\measuredangle y_{re}$
1.8-03	ENT↑	$\lvert y_{re} \rvert$
12.	GSBA	ij
-34.	ENT↑	Im y_{fe}
25.	→P	Re y_{fe}
1.-03	x	
21.	GSBA	ij
4.	ENT↑	Im y_{oe}
.3	→P	Re y_{oe}
1.-03	x	
22.	GSBA	ij
	GSBE	print stored values
49.31+00	***	$\measuredangle y_{ie}$
14.11-03	***	$\lvert y_{ie} \rvert$
-90.00+00	***	$\measuredangle y_{re}$
1.800-03	***	$\lvert y_{re} \rvert$
-53.67+00	***	$\measuredangle y_{fe}$
42.20-03	***	$\lvert y_{fe} \rvert$
85.71+00	***	$\measuredangle y_{oe}$
4.011-03	***	$\lvert y_{oe} \rvert$
	GSBB	convert CE → CB
	GSBE	print CB values
-31.45+00	***	$\measuredangle y_{ib}$
40.44-03	***	$\lvert y_{ib} \rvert$
82.23+00	***	$\measuredangle y_{rb}$
-2.220-03	***	$\lvert y_{rb} \rvert$
-49.86+00	***	$\measuredangle y_{fb}$
-39.24-03	***	$\lvert y_{fb} \rvert$
85.71+00	***	$\measuredangle y_{ob}$
4.011-03	***	$\lvert y_{ob} \rvert$

Load Program EE1-06A (EE pac)		
50.	GSBb	load reference Z & convert y parameters to s parameters
Load this program (Program 4-2)		
	GSBc	print s parameters
139.2+00	***	$\measuredangle s_{11}$
450.8-03	***	$\lvert s_{11} \rvert$
95.19+00	***	$\measuredangle s_{12}$
77.48-03	***	$\lvert s_{12} \rvert$
-36.90+00	***	$\measuredangle s_{21}$
1.369+00	***	$\lvert s_{21} \rvert$
-15.96+00	***	$\measuredangle s_{22}$
1.059+00	***	$\lvert s_{22} \rvert$
		load R_L and calc ρ_L using 50 ohm Z_o
0.	ENT↑	Im R_L
220.	ENT↑	Re R_L
50.	GSBe	Z_o
0.000+00	***	$\measuredangle \rho_L$
625.6-03	***	$\lvert \rho_L \rvert$
	GSBB	load ρ_L into program
	GSBb	execute design
-13.06+00	***	$\measuredangle s_{11}'$
1.067+00	***	$\lvert s_{11}' \rvert$
13.06+00	***	$\measuredangle 1/s_{11}'$
937.0-03	***	$\lvert 1/s_{11}' \rvert$
403.8+00	***	Im Z_L for $Z_o = 50$
116.3+00	***	Re Z_L
3.471+00	***	$Q_{min} = \text{Im } Z_L / \text{Re } Z_L$

Fig. 4-2.5 HP-97 printout for 400 MHz case.

Program Listing I

Step	Key	Comment
001	*LBLA	**LOAD S PARAMETERS:**
002	ENT↑	
003	+	
004	2	calculate storage index
005	1	
006	-	
007	STOI	
008	R↓	recover and store θ_{ij}
009	STOi	
010	ISZI	increment index
011	R↓	recover and store s_{ij}
012	STOi	
013	GTO9	goto space and return
014	*LBLB	**ENTER LOAD REFLECTION COEF**
015	STO6	store magnitude
016	X⇄Y	
017	STO7	store angle
018	GTO9	goto space and return
019	*LBLC	**CALCULATE s_{11}'**
020	RCL3	$\lvert s_{12}\rvert$
021	STO8	
022	RCL4	$\angle s_{12}$
023	STO9	
024	RCLB	$\lvert s_{21}\rvert$ $s_{11}' = s_{11} + \frac{s_{12} s_{21} \rho_L}{1 - s_{22} \rho_L}$
025	ST×8	
026	RCLC	$\angle s_{21}$
027	ST+9	
028	RCL6	$\lvert \rho_L \rvert$
029	ST×8	
030	RCL7	$\angle \rho_L$
031	ST+9	
032	RCLE	$\angle s_{22}$
033	RCL7	$\angle \rho$
034	+	
035	RCLD	$\lvert s_{22}\rvert$
036	RCL6	$\lvert \rho \rvert$
037	×	
038	CHS	
039	→R	
040	EEX	
041	+	
042	→P	$1 - s_{22} \cdot \rho_L$
043	ST÷8	
044	X⇄Y	
045	ST-9	
046	RCL9	} $\frac{s_{12} \cdot s_{21} \cdot \rho_L}{1 - s_{22} \cdot \rho_L}$ in polar coordinates
047	RCL8	
048	→R	
049	STO8	} $\frac{s_{12} \cdot s_{21} \cdot \rho_L}{1 - s_{22} \cdot \rho_L}$ in rect coordinates
050	X⇄Y	
051	STO9	
052	RCL2	s_{11}
053	RCL1	
054	→R	
055	ST+8	
056	X⇄Y	
057	ST+9	
058	RCL9	} $s_{11} - \frac{s_{12} \cdot s_{21} \cdot \rho_L}{1 - s_{22} \cdot \rho_L}$ rect
059	RCL8	
060	→P	
061	STO8	} $s_{11} - \frac{s_{12} \cdot s_{21} \cdot \rho_L}{1 - s_{22} \cdot \rho_L}$ polar
062	X⇄Y	
063	STO9	
064	GTO8	goto print subroutine
065	*LBLD	**CALCULATE $\rho \cdot s_{11}'$ GIVEN ρ**
066	P⇄S	
067	STO0	$\lvert \rho \rvert$
068	X⇄Y	
069	STO1	$\angle \rho$
070	P⇄S	
071	RCL9	$\angle s_{11}'$
072	RCL8	$\lvert s_{11}' \rvert$
073	P⇄S	
074	ST×0	$\lvert \rho \rvert \cdot \lvert s_{11}' \rvert$
075	X⇄Y	
076	ST+1	$\angle \rho + \angle s_{11}'$
077	RCL0	
078	RCL1	
079	P⇄S	
080	GTO8	goto print subroutine
081	*LBLd	**CALCULATE $1/s_{11}'$**
082	1/X	reciprocate magnitude
083	X⇄Y	
084	CHS	change sign of angle
085	GTO8	goto print subroutine
086	*LBLE	**CALC Re, Im Z GIVEN $\angle\rho$↑$\lvert\rho\rvert$↑Z_o**
087	P⇄S	
088	STO4	Z_o
089	R↓	
090	→R	
091	STO2	Re ρ $Z = Z_o \frac{1+\rho}{1-\rho}$
092	EEX	
093	+	1 + Re ρ
094	X⇄Y	
095	STO3	Im ρ
096	X⇄Y	
097	→P	$\lvert 1+\rho \rvert$
098	ST×4	} $Z_o \cdot (1+\rho)$
099	X⇄Y	
100	STO5	
101	RCL3	
102	CHS	
103	EEX	
104	RCL2	
105	-	
106	→P	$1 - \rho$
107	ST÷4	} calc Z
108	X⇄Y	
109	ST-5	
110	RCL5	

REGISTERS

0	1	2	3	4	5	6	7	8	9
	$\lvert s_{11}\rvert$	$\angle s_{11}$	$\lvert s_{12}\rvert$	$\angle s_{12}$		$\lvert \rho_L \rvert$	$\angle \rho_L$	scratch	scratch

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
scratch	scratch	Im ρ_L	Re ρ_L	Z_o, $\lvert Z_L \rvert$	scratch, $\angle Z_L$	scratch			

A	B	C	D	E	I
	$\lvert s_{21}\rvert$	$\angle s_{21}$	$\lvert s_{22}\rvert$	$\angle s_{22}$	index

Program Listing II

Step	Key	Comment
111	RCL4	
112	P⇄S	
113	→R	
114	X⇄Y	
115	GTO8	goto print subroutine
116	*LBLe	**CALC $\angle\rho$, $\lvert\rho\rvert$ GIVEN Im Z↑Re Z↑Z_o**
117	P⇄S	
118	STO4	Z_o
119	R↓	
120	STO3	Re Z $\rho = \frac{Z - Z_o}{Z + Z_o}$
121	R↓	
122	STO2	Im Z
123	R↑	
124	R↑	
125	-	Z_o - Re Z
126	→P	
127	STO5	$\lvert Z_o - Z \rvert$
128	X⇄Y	
129	STO6	$\angle (Z_o - Z)$
130	RCL2	
131	RCL3	
132	RCL4	
133	+	
134	→P	$\lvert Z_o + Z \rvert$
135	ST÷5	
136	X⇄Y	
137	ST-6	
138	RCL5	$\lvert \rho \rvert$
139	RCL6	$\angle \rho$
140	P⇄S	
141	GTO8	goto print subroutine
142	*LBLa	**INTERCHANGE PORTS 1 AND 2**
143	RCLE	
144	RCL2	
145	STOE	$\angle s_{11} \rightleftarrows \angle s_{22}$
146	X⇄Y	
147	STO2	
148	RCLD	
149	RCL1	
150	STOD	$\lvert s_{11}\rvert \rightleftarrows \lvert s_{22}\rvert$
151	X⇄Y	
152	STO1	
153	RCLC	
154	RCL4	
155	STOC	$\angle s_{12} \rightleftarrows \angle s_{21}$
156	X⇄Y	
157	STO4	
158	RCLB	
159	RCL3	
160	STOB	$\lvert s_{12}\rvert \rightleftarrows \lvert s_{21}\rvert$
161	X⇄Y	
162	STO3	
163	GTO7	goto R/S lockup
164	*LBLb	**EXECUTE fA, C, fD, 50 E**
165	GSBa	
166	GSBC	
167	GSBd	
168	5	} use 50 ohm Z_o
169	0	
170	GSBE	
171	÷	
172	GSB5	
173	GTO9	goto space and return subr
174	*LBLc	**PRINT S PARAMETER MATRIX**
175	RCL1	
176	RCL2	s_{11}
177	GSB8	
178	RCL3	
179	RCL4	s_{12}
180	GSB8	
181	RCLB	
182	RCLC	s_{21}
183	GSB8	
184	RCLD	s_{22}
185	RCLE	
186	*LBL8	print subroutine
187	GSB5	
188	X⇄Y	
189	GSB5	
190	*LBL9	space and return subroutine
191	F0?	space if flag 0 is set
192	SPC	
193	RTN	
194	*LBL7	R/S lockup subroutine
195	R/S	
196	GTO7	
197	*LBL5	print or R/S subroutine
198	F0?	if flag 0 is set, print and return
199	PRTX	
200	F0?	
201	RTN	
202	R/S	otherwise stop
203	RTN	

Notes:

Flag 0 controls the print or R/S decision. It should be set or reset to reflect the users choice of printed output, or program halts for output respectively at the time the magnetic card is recorded.

LABELS

A	B	C	D	E
load s parameters	enter load reflection coef	calculate s_{11}'	calculate $s_{11}' \cdot \rho$	$\rho, Z_o \to Z$
a interchange ports 1 & 2	b fA, C, fD, E	c print s matrix	d calculate $1/s_{11}'$	e $Z, Z_o \to \rho$
0	1	2	3	4
5 print or R/S subroutine	6	7 R/S lockup	8 print & space subroutine	9 space subroutine

FLAGS

Flag	Use
0	print
1	
2	
3	

SET STATUS

FLAGS (ON / OFF)	TRIG	DISP
0 ☐ ☐	DEG ■	FIX
1	GRAD	SCI
2	RAD	ENG ■
3		n 3

PROGRAM 4-3 TRANSISTOR CONFIGURATION CONVERSION.

Program Description and Equations Used

This program allows conversion between common emitter, common base, and common collector configurations of transistor h parameters or y parameters, as well as conversions between the h and the y parameters.

The configuration conversions is done by operating on the y parameters and converting to and from the h parameters for data input and output. To make the program operate in either h or y parameters, the conversion process is skipped for the y parameter case. Label 7 of the program contains the coding that accomplishes the h to y, or y to h conversion. Label 7 is called at the beginning and end of the configuration conversion, and flag 0 is used to indicate whether or not the subroutine under label 7 should be skipped or not.

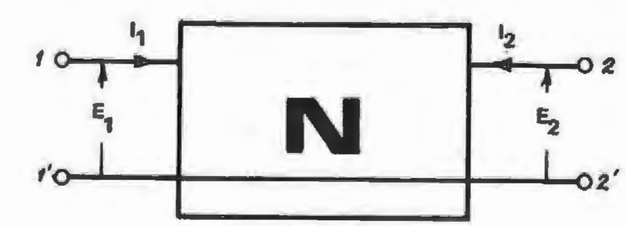

Figure 4-3.1 Two-port network conventions.

Given a two-port network with port voltages and currents as defined by Fig. 4-3.1, the y and h parameters are defined as follows:

h parameters

$$\begin{bmatrix} E_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ E_2 \end{bmatrix} \qquad (4\text{-}3.1)$$

y parameters

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad (4\text{-}3.2)$$

The network ports correspondence to the transitor elements is shown in Table 4-3.1.

Table 4-3.1 Transistor 2-port correspondences

Configuration	1	1' or 2'	2
CE	B	E	C
CB	E	B	C
CC	B	C	E

The h parameters are converted to y parameters with the following transformation [15]:

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \frac{1}{h_{11}} \begin{bmatrix} 1, & -h_{12} \\ h_{21}, & h_{11}h_{22} - h_{12}h_{21} \end{bmatrix} \quad (4\text{-}3.3)$$

Likewise, the y parameters are converted to h parameters in similar fashion:

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \frac{1}{y_{11}} \cdot \begin{bmatrix} 1, & -y_{12} \\ y_{21}, & y_{11}y_{22} - y_{21}y_{12} \end{bmatrix} \quad (4\text{-}3.4)$$

Since the form of both conversions is identical, the same subroutine is used for both conversions (subroutine 7).

The y matrix representing the present transistor configuration is

transformed into another y matrix representing the new transistor configuration. This new matrix is designated y' for clarity. These transformations are:

For CE → CB or CB → CE,

$$\begin{bmatrix} y_{11}' & y_{12}' \\ y_{21}' & y_{22}' \end{bmatrix} = \begin{bmatrix} \{y_{11} + y_{22} + y_{12} + y_{21}\} & \{-(y_{12} + y_{22})\} \\ \{-(y_{21} + y_{22})\} & \{\ y_{11}\ \} \end{bmatrix} \quad (4\text{-}3.5)$$

For CC → CE or CE → CC,

$$\begin{bmatrix} y_{11}' & y_{12}' \\ y_{21}' & y_{22}' \end{bmatrix} = \begin{bmatrix} \{\ y_{11}\ \} & \{\ -(y_{11} + y_{12})\ \} \\ \{-(y_{11} + y_{21})\} & \{y_{11} + y_{22} + y_{21} + y_{12}\} \end{bmatrix} \quad (4\text{-}3.6)$$

For CC → CB,

$$\begin{bmatrix} y_{11}' & y_{12}' \\ y_{21}' & y_{22}' \end{bmatrix} = \begin{bmatrix} \{\ y_{22}\ \} & \{\ -(y_{21} + y_{22})\ \} \\ \{-(y_{12} + y_{22})\} & \{y_{11} + y_{12} + y_{21} + y_{22}\} \end{bmatrix} \quad (4\text{-}3.7)$$

For CB → CC,

$$\begin{bmatrix} y_{11}' & y_{12}' \\ y_{21}' & y_{22}' \end{bmatrix} = \begin{bmatrix} \{y_{11} + y_{22} + y_{21} + y_{12}\} & \{-(y_{11} + y_{21})\ \} \\ \{-(y_{11} + y_{12})\} & \{\ y_{11}\ \} \end{bmatrix} \quad (4\text{-}3.8)$$

After the respective conversion is complete, the y' matrix has replaced the y matrix in storage.

In looking over the various conversions, one will notice similarities in the operations used. There are four basic operations used to perform all the conversions:

1) no change;

2) (y_{11} or y_{22}) + y_{12};

3) (y_{11} or y_{22}) + y_{21}; and

4) $y_{11} + y_{22} + y_{12} + y_{21}$.

The choice between y_{11} and y_{22} or y_{12} and y_{22} can be taken care of by interchanging the appropriate y matrix elements prior to these calculations. This matrix reordering is accomplished under label 3. The matrix conversion calculation is done under label 6 (two places); thus, these subroutines are selectively used to achieve all conversions.

4-3 **User Instructions**

TRANSISTOR CONFIGURATION CONVERSION				
	CB → CE	CB → CC	CE → CC	h or y ? 0 or 1
load θ_{ij}↑ $\|\|_{ij}$↑ij	CE → CB	CC → CB	CC → CE	print matrix

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Select h or y matrix mode			
	a) h parameters	0	f E	0
	b) y parameters	1	f E	1
3	Load matrix to be converted			
	a) angle of h_{ij} or y_{ij}	θ_{ij}	ENT↑	
	b) magnitude of h_{ij} or y_{ij}	$\| \|_{ij}$	ENT↑	
	c) subscript	ij	A	
	repeat this step for ij = 11, 12, 21, 22			
	in any order			
4	Select conversion desired			
	a) common emitter to common base		B	
	b) common base to common emitter		f B	
	c) common collector to common base		C	
	d) common base to common collector		f C	
	e) common collector to common emitter		D	
	f) common emitter to common collector		f D	
5	Print converted matrix		E	θ_{11}
				$\| \|_{11}$
				θ_{12}
				$\| \|_{12}$
				θ_{21}
				$\| \|_{21}$
				θ_{22}
				$\| \|_{22}$

Example 4-3.1

Convert the following common collector h parameter matrix to a common base h parameter matrix:

$$\begin{bmatrix} h_{ic} & h_{rc} \\ h_{fc} & h_{oc} \end{bmatrix} = \begin{bmatrix} 1000 \angle 40^\circ & 10^{-4} \angle -50^\circ \\ 100 \angle 40^\circ & 50 \times 10^{-6} \angle 0 \end{bmatrix}$$

PROGRAM INPUT			PROGRAM OUTPUT		
40.	ENT↑	$\angle h_{ic}$	0.	GSBE	select h parameters
1000.	ENT↑	$\lvert h_{ic} \rvert$		GSBC	execute CC → CB conv
11.	GSBA	ij		GSBE	print stored matrix
-50.	ENT↑	$\angle h_{rc}$	-9.971+00	***	$\angle h_{ib}$
1.-04	ENT↑	$\lvert h_{rc} \rvert$	22.60+03	***	$\lvert h_{ib} \rvert$
12.	GSBA	ij	-9.967+00	***	$\angle h_{rb}$
40.	ENT↑	$\angle h_{fc}$	2.261+03	***	$\lvert h_{rb} \rvert$
100.	ENT↑	$\lvert h_{fc} \rvert$	-179.9+00	***	$\angle h_{fb}$
21.	GSBA	ij	1.000+00	***	$\lvert h_{fb} \rvert$
0.	ENT↑	$\angle h_{oc}$	-49.97+00	***	$\angle h_{oc}$
50.-06	ENT↑	$\lvert h_{oc} \rvert$	1.130-03	***	$\lvert h_{oc} \rvert$
22.	GSBA	ij			

Common base h parameter matrix from HP-97 output:

$$\begin{bmatrix} h_{ib} & h_{rb} \\ h_{fb} & h_{ob} \end{bmatrix} = \begin{bmatrix} 22600 \angle -9.971^\circ & 2261 \angle -9.967^\circ \\ 1.000 \angle -179.9^\circ & 1.130 \times 10^{-3} \angle -49.97^\circ \end{bmatrix}$$

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD MATRIX ELEMENTS,
002	ENT↑	calculate storage index from subscript
003	+	
004	2	
005	1	
006	-	
007	STOI	
008	R↓	recover and store $\lvert\ \rvert_{ij}$
009	STOi	
010	ISZI	increment storage index
011	R↓	recover and store θ_{ij}
012	STOi	
013	GTO4	return control to keyboard
014	*LBLC	CONVERT CC → CB PARAMETERS
015	GSB6	take [h] → [y] → [y']
016	*LBL3	reorder matrix elements
017	RCL1	$\lvert\ \rvert_{11} \rightleftarrows \lvert\ \rvert_{22}$
018	RCLD	
019	STO1	
020	R↓	
021	STOD	
022	RCL3	$\lvert\ \rvert_{12} \rightleftarrows \lvert\ \rvert_{21}$
023	RCLB	
024	STO3	
025	R↓	
026	STOB	
027	RCL2	$\theta_{11} \rightleftarrows \theta_{22}$
028	RCLE	
029	STO2	
030	R↓	
031	STOE	
032	RCL4	$\theta_{12} \rightleftarrows \theta_{21}$
033	RCLC	
034	STO4	
035	R↓	
036	STOC	
037	GTO7	convert y' h
038	*LBLb	CONVERT OB → CE PARAMETERS
039	*LBLB	CONVERT CE → CB PARAMETERS
040	GSB6	take [h] → [y] → [y']
041	GTO7	convert [y'] → [h]
042	*LBL6	
043	GSB7	convert h params to y params
044	RCLC	calculate and store:
045	RCLB	
046	RCLE	
047	RCLD	
048	GSB8	$y_{21}' = -(y_{21} + y_{22})$
049	CHS	
050	STOB	
051	R↓	
052	STOC	
053	RCL4	calculate and store:
054	RCL3	
055	RCLE	
056	RCLD	
057	GSB8	
058	CHS	$y_{12}' = -(y_{12} + y_{22})$
059	STO3	
060	X⇄Y	
061	STO4	
062	X⇄Y	calculate and store:
063	RCLC	
064	RCLB	
065	GSB8	
066	RCLE	
067	RCLD	
068	GSB8	
069	CHS	$y_{11}' = -y_{22} + (y_{21}' + y_{12}') + y_{11}$
070	RCL2	
071	RCL1	$= y_{11} + y_{12} + y_{21} + y_{22}$
072	GSB8	
073	STO1	
074	R↓	
075	STO2	
076	RTN	
077	*LBLd	CONVERT CE → CC PARAMETERS
078	*LBLD	CONVERT CC → CE PARAMETERS
079	GSB6	take [h] → [y] → [y']
080	GTO7	convert [y'] → [h]
081	*LBLc	CONVERT CB → CC PARAMETERS
082	GSB6	take [h] → [y] → [y']
083	GTO3	reorder matrix elements
084	*LBL6	
085	GSB7	transform [h] → [y]
086	RCL2	calculate and store:
087	RCL1	
088	RCL4	
089	RCL3	
090	GSB8	
091	CHS	$y_{12}' = -(y_{11} + y_{12})$
092	STO3	
093	R↓	
094	STO4	
095	RCL2	calculate and store:
096	RCL1	
097	RCLC	
098	RCLB	
099	GSB8	
100	CHS	$y_{21}' = -(y_{11} + y_{21})$
101	STOB	
102	X⇄Y	
103	STOC	
104	X⇄Y	calculate and store:
105	RCL4	
106	RCL3	
107	GSB8	
108	RCL2	$y_{22}' = y_{11} + y_{12} + y_{21} + y_{22}$
109	RCL1	
110	GSB8	

REGISTERS

0	1	2	3	4	5	6	7	8	9
	$\lvert\ \rvert_{11}$	θ_{11}	$\lvert\ \rvert_{12}$	θ_{12}		$\lvert\Delta\rvert$	$\angle\Delta$		

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
	$\lvert\ \rvert_{21}$	θ_{21}	$\lvert\ \rvert_{22}$	θ_{22}	index

4-3 Program Listing II

```
   CHS
  RCLE
  RCLD
  GSB8
  STOD
    R↓
  STOE
   RTN    return to main program
 *LBL7    subroutine to convert [y]⇄[h]
   F0?    if flag 0 is set
   RTN
  RCL2
  RCL1    given:
  RCLD
  RCLE    | a11  a12 |
  GSB9    |          | = A
  ST06    | a21  a22 |
    R↓
  ST07
  RCL4    calculate and store the
  RCL3    determinant of A:
  RCLB
  RCLC    ΔA = a11·a22 - a21·a12
  GSB9
   CHS
  RCL7
  RCL6
  GSB8
  ST06
    R↓
  ST07
  RCL2    calculate and store:
   CHS
  ST02
  RCL1    a11' = 1 / a11
   1/X
  ST01
  RCL3
   CHS    calculate and store:
  RCL4
  GSB9    a12' = - a12 / a11
  ST03
    R↓
  ST04
  RCL2
  RCL1    calculate and store:
  RCLB
  RCLC
  GSB9    a21' = - a21 / a11
  STOB
    R↓
  STOC
  RCL2
  RCL1
  RCL6
  RCL7    calculate and store:
  GSB9
  STOD
    R↓    a22' = ΔA / a11
  STOE
   RTN    return to subroutine call
  GT02    goto R/S lockup
 *LBL8    complex addition subroutine
    →R
    R↓    input and output are in
    R↓    polar co-ordinates
    →R
   X⇄Y
    R↓
     +
    R↓
     +
    R↑
    →P
   RTN
 *LBL9    complex multiply subroutine
    R↓
     ×    input and output are in
    R↓    polar co-ordinates
     +
    R↑
    →R
    →P
   RTN
 *LBLE    PRINT STORED MATRIX
  RCL1
  RCL2
  GSB5
  RCL3
  RCL4
  GSB5
  RCLB
  RCLC
  GSB5
  RCLD
  RCLE
 *LBL5    print subroutine
  PRTX -- or R/S
   X⇄Y
  PRTX -- or R/S
 *LBL4    space and return subroutine
   SPC
   RTN
 *LBL2    R/S lockup subroutine
   R/S
  GT02
 *LBLe    SELECT y OR h PARAMETERS
   CF0
  X>0?    set flag 0 if "1" entered
   SF0
   RTN    return to keyboard control
```

LABELS					FLAGS
A load data	B CE→CB	C CC→CB	D CC→CE	E print matrix	0 set for y
a	b CB→CE	c CB→CC	d CE→CC	e select y or h	1
0	1	2	3 rearrange matrix	4 space & rtn subroutine	2
5 print subroutine	6 [h]→[h']	7 [h]⇄[y]	8 complex add	9 complex multiply	3

SET STATUS				
FLAGS	ON	OFF	TRIG	DISP
0	■		DEG ■	FIX
1			GRAD	SCI
2			RAD	ENG ■
3				n 3

PROGRAM 4-4 COMPLEX 2x2 MATRIX OPERATIONS – PART 1.

Program Description and Equations Used

This program is one of two programs to manipulate complex 2x2 matrices. When dealing with high frequency amplifiers employing feedback, and input and output networks, one way of obtaining the overall amplifier response is to operate on the matrices that describe these 2-port networks. Shunt feedback may be included within the transistor transfer matrix through Y matrix addition. Y matrices can be converted to Z matrices using the complex matrix inverse routine. Series feedback is included by adding Z matrices. The input and output networks are included by multiplying ABCD (transmission) matrices.

This program will perform matrix addition ($A + B \rightarrow A$), subtraction ($A - B \rightarrow A$), multiplication ($AB \rightarrow A$), and interchange ($A \rightleftarrows B$) with 2x2 matrices having complex coefficients. Data entry and output may be in either rectangular or polar format. All data stored and used by the program is in rectangular format. If flag 1 is set, polar format is indicated and the data is converted to and from rectangular format upon data input or output respectively.

The program operation is very straightforward, and matrix operations are done in the conventional manner. Two subroutines are used, one for complex addition and the other for complex multiplication. See [6], [14] for matrix algebra details.

Both this program and the companion program (Program 4-5) share common register storage allocations; thus, matrix manipulations requiring functions contained in different programs are easily accommodated.

Matrix addition and subtraction:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$r_{11} = a_{11} \pm b_{11}$$
$$r_{22} = a_{22} \pm b_{22}$$
$$r_{21} = a_{21} \pm b_{21}$$
$$r_{22} = a_{22} \pm b_{22}$$

The R matrix replaces the A matrix at the completion of the routine.

Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$r_{11} = a_{11}\, b_{11} + a_{12}\, b_{21}$$
$$r_{12} = a_{11}\, b_{12} + a_{12}\, b_{22}$$
$$r_{21} = a_{21}\, b_{11} + a_{22}\, b_{21}$$
$$r_{22} = a_{21}\, b_{12} + a_{22}\, b_{22}$$

Again, the R matrix replaces the A matrix at the completion of the routine.

Matrix interchange:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightleftarrows \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad \begin{matrix} a_{11} \rightleftarrows b_{11} \\ a_{12} \rightleftarrows b_{12} \\ a_{21} \rightleftarrows b_{21} \\ a_{22} \rightleftarrows b_{22} \end{matrix}$$

User Instructions

COMPLEX 2x2 MATRIX OPERATIONS - PART 1				
print A	print B	rect/polar 0 / 1	A ⇄ B	
load A	load B	A + B → A	A - B → A	A x B → A

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of the program card			
2	select polar or rectangular format		f 0	0 (rect)
			f 0	1 (polar)
			f 0	0 (rect)
				⋮
3	Load A matrix in selected format (step 2)			
	rectangular format shown here			
	a) imaginary part of matrix element	Im a_{ij}	ENT↑	
	b) real part of matrix element	Re a_{ij}	ENT↑	
	c) load element subscript	ij	A	
	Do this step for subscripts 11, 12, 21, 22			
	in any order.			
4	Load B matrix in selected format (step 2)			
	polar format shown here			
	a) load angle of matrix element	$\measuredangle b_{ij}$	ENT↑	
	b) load magnitude of matrix element	$\lvert b_{ij} \rvert$	ENT↑	
	c) load element subscript	ij	B	
	Do this step for subscripts 11, 12, 21, 22			
	in any order			
5	To print matrices in chosen format (say polar)			
	a) A matrix -- use f A		f *	$\measuredangle_{11}$
	b) B matrix -- use f B			$\lvert\ \rvert_{11}$
				$\measuredangle_{12}$
				$\lvert\ \rvert_{12}$
				$\measuredangle_{21}$
				$\lvert\ \rvert_{21}$
				$\measuredangle_{22}$
				$\lvert\ \rvert_{22}$

4-4 # User Instructions

COMPLEX 2x2 MATRIX OPERATIONS - PART 1

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
6	To add matrices A and B with result replacing A (use step 5 to print result)		[C]	
7	To subtract matrices A and B with result replacing A (use step 5 to print results)		[D]	
8	To multiply matrices A and B with result replacing A (use step 5 to print results)		[E]	
9	To interchange matrices A and B (A⇄B) (use step 5 to print results)		[f] [D]	
	General notes			
	1) Matrix data and operations are stored and manipulated in rectangular format and converted to and from polar for data input and output if flag 1 is set.			
	2) After any operation or input, the presently stored matrices can be recorded on a magnetic card using the WDATA command, and later re-entered into storage.			

Example 4-4.1

Given

$$A = \begin{bmatrix} (3 + j4) & (4 + j5) \\ (5 + j6) & (2 + j4) \end{bmatrix}, \qquad B = \begin{bmatrix} (4 + j5) & (5 + j6) \\ (6 + j7) & (7 + j8) \end{bmatrix}$$

Load the above matrices, store them on a data card, then perform A + B, A − B, and A x B. The HP-97 printout for the matrix loading is shown below, and the program output is shown on the next page. The B matrix is loaded in scrambled order to demonstrate the free form loading feature of the program.

HP-97 PRINTOUT FOR EXAMPLE 4-4.1 INPUT

A MATRIX LOADING			B MATRIX LOADING		
4.00	ENT↑	Im a_{11}	6.00	ENT↑	Im b_{12}
3.00	ENT↑	Re a_{11}	5.00	ENT↑	Re b_{12}
11.00	GSBA	ij	12.00	GSBB	ij
5.00	ENT↑	Im a_{12}	7.00	ENT↑	Im b_{21}
4.00	ENT↑	Re a_{12}	6.00	ENT↑	Re b_{21}
12.00	GSBA	ij	21.00	GSBB	ij
6.00	ENT↑	Im a_{21}	8.00	ENT↑	Im b_{22}
5.00	ENT↑	Re a_{21}	7.00	ENT↑	Re b_{22}
21.00	GSBA	ij	22.00	GSBB	ij
4.00	ENT↑	Im a_{22}	5.00	ENT↑	Im b_{11}
2.00	ENT↑	Re a_{22}	4.00	ENT↑	Re b_{11}
22.00	GSBA	ij	11.00	GSBB	ij
				WDTA	record data card

HP-97 PRINTOUT FOR EXAMPLE 4-4.1 OUTPUT

	GSBa	print A matrix
4.00	***	Im a_{11}
3.00	***	Re a_{11}
5.00	***	Im a_{12}
4.00	***	Re a_{12}
6.00	***	Im a_{21}
5.00	***	Re a_{21}
4.00	***	Im a_{22}
2.00	***	Re a_{22}
	GSBb	print B matrix
5.00	***	Im b_{11}
4.00	***	Re b_{11}
6.00	***	Im b_{12}
5.00	***	Re b_{12}
7.00	***	Im b_{21}
6.00	***	Re b_{21}
8.00	***	Im b_{22}
7.00	***	Re b_{22}

	GSBC	execute matrix addition
	GSBa	print resultant matrix
9.00	***	Im r_{11}
7.00	***	Re r_{11}
11.00	***	Im r_{12}
9.00	***	Re r_{12}
13.00	***	Im r_{21}
11.00	***	Re r_{21}
12.00	***	Im r_{22}
9.00	***	Re r_{22}

Note that the R matrix has replaced the A matrix in storage.

Reload A and B matrices by reading data card.

	GSBD	execute mat subtraction
	GSBa	print resultant matrix
-1.00	***	Im r_{11}
-1.00	***	Re r_{11}
-1.00	***	Im r_{12}
-1.00	***	Re r_{12}
-1.00	***	Im r_{21}
-1.00	***	Re r_{21}
-4.00	***	Im r_{22}
-5.00	***	Re r_{22}

Reload A and B matrices by reading data card.

	GSBE	exec mat multiplication
	GSBa	print resultant matrix
89.00	***	Im r_{11}
-19.00	***	Re r_{11}
105.00	***	Im r_{12}
-21.00	***	Re r_{12}
87.00	***	Im r_{21}
-26.00	***	Re r_{21}
104.00	***	Im r_{22}
-29.00	***	Re r_{22}

Example 4-4.2

Because the resultant matrix replaces the A matrix in storage, operations may be chained. This example demonstrates that chaining ability starting with the A and B matrices given in Example 4-4.1.

	GSBE	execute matrix multiplication: A x B → A
	GSBC	execute matrix addition: AB + B → A
	GSBd	execute matrix interchange: AB + B ⇄ B
	GSBE	execute matrix multiplication: B(AB + B)→ A
	GSBa	print resultant A matrix
651.00	***	Im a_{11}
-1194.00	***	Re a_{11}
792.00	***	Im a_{12}
-1401.00	***	Re a_{12}
957.00	***	Im a_{21}
-1640.00	***	Re a_{21}
1162.00	***	Im a_{22}
-1923.00	***	Re a_{22}

The same data can be outputted (printed) in polar format using the polar-rectangular toggle under label "c" to bring a 1 to the display.

	GSBc GSBc	} use polar-rectangular selection toggle
	GSBa	print A matrix in polar format
151.40	***	$\angle a_{11}$
1359.94	***	$\lvert a_{11}\rvert$
150.52	***	$\angle a_{12}$
1609.37	***	$\lvert a_{12}\rvert$
149.73	***	$\angle a_{21}$
1898.80	***	$\lvert a_{21}\rvert$
148.86	***	$\angle a_{22}$
2246.81	***	$\lvert a_{22}\rvert$

Program Listing I

*LBLA	LOAD MATRIX A
SF2	indicate matrix A
*LBLB	LOAD MATRIX B
1	calculate storage register location from subscript
2	
-	
X>0?	
8	
X>0?	
-	
ENT↑	
+	
3	
+	
EEX	
F2?	
CLX	
+	
R↓	if polar data, convert to rectangular format
F1?	
→R	
R↑	recover storage index
GSB8	store matrix element
GTO1	goto space and return
*LBLa	PRINT MATRIX A
EEX	initialize index register for matrix A
STOI	
GTOe	jump
*LBLb	PRINT MATRIX B
2	initialize index register for matrix B
STOI	
*LBLe	matrix print subroutine
GSB4	recall matrix element
F1?	convert to polar format if flag 1 is set
→P	
X⇄Y	print matrix element as complex quantity (may be R/S statements if desired)
PRTX	
X⇄Y	
PRTX	
SPC	
ISZI	increment index by 2
ISZI	
8	test for loop exit
RCLI	
X≤Y?	
GTOe	
GTO1	goto space and return
*LBLC	ADD A AND B MATRICES
CF0	indicate matrix addition
GTOC	jump
*LBLD	SUBTRACT A AND B MATRICES
SF0	indicate matrix subtraction
*LBLC	initialize index register
8	
STOI	
*LBL0	matrix add/subtract subr
GSB4	recall matrix B element
F0?	change sign of element parts if subtraction is indicated
CHS	
X⇄Y	
F0?	
CHS	
X⇄Y	
DSZI	decrement index
GSB4	recall matrix A element
GSB2	perform complex addition
GSB5	store result as matrix A
DSZI	decrement index and test for loop exit
GTO0	
GTO1	goto space and return
*LBLE	MATRIX MULTIPLICATION
1	calculate and temporarily store: $a_{11}\cdot b_{11} + a_{12}\cdot b_{21} = r_{11}$
2	
SF2	
GSB7	
3	
6	
GSB7	
9	
GSB8	
1	calculate and temporarily store: $a_{11}\cdot b_{12} + a_{12}\cdot b_{22} = r_{12}$
4	
SF2	
GSB7	
3	
8	
GSB7	
STOA	
X⇄Y	
STOB	
2	calculate and temporarily store: $a_{21}\cdot b_{11} + a_{22}\cdot b_{21} = r_{21}$
5	
SF2	
GSB7	
7	
6	
GSB7	
STOC	
X⇄Y	
STOD	
4	calculate and store: $a_{21}\cdot b_{12} + a_{22}\cdot b_{22} = r_{22}$
5	
SF2	
GSB7	
7	
8	
GSB7	
7	
GSB8	

REGISTERS

0 scratch	1 Re a_{11}	2 Re b_{11}	3 Re a_{12}	4 Re b_{12}	5 Re a_{21}	6 Re b_{21}	7 Re a_{22}	8 Re b_{22}	9 temp Re r_{11}
S0 scratch	S1 Im a_{11}	S2 Im b_{11}	S3 Im a_{12}	S4 Im b_{12}	S5 Im a_{21}	S6 Im b_{21}	S7 Im a_{22}	S8 Im b_{22}	S9 temp Im r_{11}

A temporary Re r_{12}	B temporary Im r_{12}	C temporary Re r_{21}	D temporary Im R_{21}	E scratchpad	I index

Program Listing II

9	$r_{11} \rightarrow a_{11}$
GSB9	
EEX	
GSB8	
RCLB	$r_{12} \rightarrow a_{12}$
RCLA	
3	
GSB8	
RCLD	$r_{21} \rightarrow a_{21}$
RCLC	
5	
GSB8	
*LBL1	space and return subroutine
SPC	
RTN	
*LBL7	matrix multiply subroutine
EEX	recall first matrix element
1	
÷	
GSB9	
RCLI	recall second matrix element
FRC	
EEX	
1	
x	
GSB9	
STOE	complex multiplication
R↓	
STOI	
R↓	
ENT↑	
R↑	
x	
R↑	
RCLI	
X⇄Y	
x	
LSTX	
R↓	
-	
R↑	
RCLE	
x	
R↑	
RCLI	
x	
+	
X⇄Y	
F2?	jump if first product
GTO7	
0	recall first product from scratchpad storage
GSB9	
*LBL2	complex add subroutine
X⇄Y	
R↓	
+	
R↓	
+	
R↑	
RTN	
*LBL7	setup scratchpad index
0	
*LBL8	store storage index subr
STOI	
R↓	
*LBL5	complex storage subroutine
STOi	
X⇄Y	
P⇄S	
STOi	
P⇄S	
RTN	
*LBL9	store recall index subr
STOI	
R↓	
*LBL4	complex recall subroutine
P⇄S	
RCLi	
P⇄S	
RCLi	
RTN	
*LBLc	POLAR/RECTANGULAR TOGGLE
CF1	clear flag 1 to indicate rectangular format and place a zero in the display
CLX	
RTN	
*LBLc	
SF1	set flag 1 to indicate polar format and place a one in the display
EEX	
RTN	
*LBLd	MATRIX INTERCHANGE
8	initialize index
STOI	
*LBL6	
GSB4	recall corresponding matrix elements
DSZI	
GSB4	
ISZI	
GSB5	interchange and store corresponding elements
DSZI	
R↓	
R↓	
GSB5	
DSZI	decrement index and test for loop exit
GTO6	
GTO1	goto space and return

LABELS					FLAGS
A load A	B load B	C A+B→A	D A−B→A	E A×B→A	0 subtract
a print A	b print B	c polar/rect 1 / 0	d A⇄B	e print loop start	1 polar
0 matrix add/subtract	1 space & rtn	2 complex addition	3	4 complex recall	2 don't continue summation
5 complex store	6 A⇄B subroutine	7 matrix multiplication	8 store index & complex sto	9 store index & complex rcl	3

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0		■	DEG	FIX
1		■	GRAD	SCI
2		■	RAD	ENG ■
3				n 3

PROGRAM 4-5 COMPLEX 2x2 MATRIX OPERATIONS – PART 2.

Program Description and Equations Used

This program is the second of two programs to manipulate complex 2x2 matrices. This program will perform matrix inverse ($A^{-1} \rightarrow A$), matrix transpose ($A^T \rightarrow A$), matrix complex conjugate ($A^* \rightarrow A$), and matrix interchange ($A \rightleftarrows B$). Because the resultant matrix from the matrix operation replaces the A matrix, chaining of matrix operations without data re-entry is easily done.

This program shares common register storage with Program 4-4, hence, matrix operations that require concatenation of routines contained in two different programs can be done without reloading any previous data.

The user may elect to work in either the polar or the rectangular co-ordinate systems, however, all data is stored in rectangular format. If flag 1 is set, the input data is converted from polar to rectangular, and vice-versa for output.

The algorithms used are:

Matrix inverse:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad (4\text{-}5.1)$$

where $|A|$ is the determinant of A,

$$|A| = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \qquad (4\text{-}5.2)$$

Matrix transpose:

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \qquad (4\text{-}5.3)$$

Matrix complex conjugate:

$$A^* = \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix}$$

Matrix interchange, see Eq. (4-4.3).

4-5 # User Instructions

COMPLEX 2x2 MATRIX OPERATIONS - PART 2				
print A	print B	polar/rect 1/0	A ⇄ B	calculate |A|
load A	load B	$A^{-1} \to A$	$A^T \to A$	$A^* \to A$

2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of the program card			
2	Select polar or rectangluar format		f C	0 (rect)
			f C	1 (polar)
			f C	0 (rect)
				⋮
3	Load matrix A in selected format (rect shown)			
	a) load imaginary part of matrix element	Im a_{ij}	ENT↑	
	b) load real part of matrix element	Re a_{ij}	ENT↑	
	c) load subscript of matrix element	ij	A	
	Repeat this step for ij 11, 12, 21, 22 in any order.			
4	Load matrix B in selected format (polar used)			
	a) load angle of matrix element	$\measuredangle b_{ij}$	ENT↑	
	b) load magnitude of matrix element	$\|b_{ij}\|$	ENT↑	
	c) load subscript of matrix element	ij	B	
	Repeat this step for ij 11, 12, 21, 22 in any order.			
5	To print matrices in chosen format (say polar)			
	a) A matrix — use f A		f *	$\measuredangle_{11}$
	b) B matrix — use f B			$\|\ \|_{11}$
				$\measuredangle_{12}$
				$\|\ \|_{12}$
				$\measuredangle_{21}$
				$\|\ \|_{21}$
				$\measuredangle_{22}$
				$\|\ \|_{22}$

4-5 **User Instructions**

COMPLEX 2x2 MATRIX OPERATIONS - PART 2

CONTINUED

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
6	To calculate matrix A inverse (use step 5 to print out resultant A matrix)		[C]	
7	To calculate matrix A transpose (use step 5 to print out resultant A matrix)		[D]	
8	To calculate matrix A complex conjugate (use step 5 to print out resultant A matrix)		[E]	
9	To calculate the determinant of the A matrix		[f] [E]	Im |A| Re |A|
10	To interchange matrices A and B in storage (use step 5 to print matrices)		[f] [D]	

Example 4-5.1

Given the A and B matrices of Example 4-4.1, calculate $B^{-1}AB$. The loading of the A and B matrices is shown in Example 4-4.1, and is omitted here for brevity (they were actually loaded from the magnetic data card from Program 4-4).

Load Program 4-4, and load A and B matrices

GSBE form AB → A

Load this program (Program 4-5)

GSBa interchange AB and B

GSBC form B^{-1}→A

Reload Program 4-4

GSBE form $B^{-1}AB$ → A

```
          GSBa  print result
-48.25    ***   Im a11
-46.00    ***   Re a11

-54.50    ***   Im a12
-55.25    ***   Re a12

 49.50    ***   Im a21
 45.75    ***   Re a21

 56.25    ***   Im a22
 53.00    ***   Re a22
```

Program Listing I

Step	Key	Comments
001	*LBLA	LOAD MATRIX A
002	SF2	indicate matrix A
003	*LBLB	LOAD MATRIX B
004	1	calculate storage register location from subscript
005	2	
006	-	
007	X>0?	
008	8	
009	X>0?	
010	-	
011	ENT↑	
012	+	
013	3	
014	+	
015	EEX	
016	F2?	
017	CLX	
018	+	
019	R↓	if polar data, convert to rectangular format
020	F1?	
021	→R	
022	R↑	recover storage index
023	GSB8	store matrix element
024	GTO1	goto space and return subr
025	*LBLa	PRINT MATRIX A
026	EEX	initialize index register for matrix A
027	STOI	
028	GTO7	jump
029	*LBLb	PRINT MATRIX B
030	2	initialize index register for matrix B
031	STOI	
032	*LBL7	matrix print subroutine
033	GSB4	recall matrix element
034	GSB0	print matrix element
035	ISZI	increment index by 2
036	ISZI	
037	8	test for loop exit
038	RCLI	
039	X≤Y?	
040	GTO7	
041	GTO1	goto space and return subr
042	*LBLe	CALCULATE DETERMINANT OF A
043	SF2	indicate determinant calc
044	*LBLC	CALCULATE A MATRIX INVERSE
045	EEX	calculate and scratchpad store $a_{11} \cdot a_{22}$
046	GSB9	
047	7	
048	GSB9	
049	GSB3	
050	9	
051	GSB8	
052	3	calculate $a_{21} \cdot a_{12}$ and subtract from $a_{11} \cdot a_{22}$ which is stored
053	GSB9	
054	5	
055	GSBe	
056	9	
057	GSB9	
058	GSB2	
059	F2?	if determinant calculation goto print routine
060	GTO0	
061	→P	calculate and store $\frac{1}{\lvert A \rvert}$
062	1/X	
063	X⇄Y	
064	CHS	
065	X⇄Y	
066	→R	
067	9	
068	GSB8	
069	3	calculate and store: $-\frac{a_{12}}{\lvert A \rvert} \rightarrow a_{12}$
070	GSB9	
071	9	
072	GSBe	
073	3	
074	GSB8	
075	5	calculate and store; $-\frac{a_{21}}{\lvert A \rvert} \rightarrow a_{12}$
076	GSB9	
077	9	
078	GSBe	
079	5	
080	GSB8	
081	EEX	calculate and store: $\frac{a_{22}}{\lvert A \rvert} \rightarrow a_{11}$
082	GSB9	
083	STOA	
084	X⇄Y	
085	STOB	
086	7	
087	GSB9	
088	9	
089	GSB9	
090	GSB3	
091	EEX	
092	GSB8	
093	RCLB	calculate and store: $\frac{a_{11}}{\lvert A \rvert} \rightarrow a_{22}$
094	RCLA	
095	9	
096	GSB9	
097	GSB3	
098	7	
099	GSB8	
100	GTO1	
101	*LBLe	multiply and change sign subroutine
102	GSB9	
103	GSB3	
104	CHS	
105	X⇄Y	
106	CHS	
107	X⇄Y	
108	RTN	

REGISTERS

0	1	2	3	4	5	6	7	8	9
scratchpad	Re a_{11}	Re b_{11}	Re a_{12}	Re b_{12}	Re a_{21}	Re b_{21}	Re a_{22}	Re b_{22}	Re $\frac{1}{\lvert A \rvert}$
S0 scratchpad	S1 Im a_{11}	S2 Im b_{11}	S3 Im a_{12}	S4 Im b_{12}	S5 Im a_{21}	S6 Im b_{21}	S7 Im a_{22}	S8 Im b_{22}	S9 Im $\frac{1}{\lvert A \rvert}$

A	B	C	D	E	I
scratchpad	scratchpad			scratchpad	index / scratch

4-5

Program Listing II

Step	Key	Comments
109	*LBL0	common output subroutine
110	F1?	convert to polar if required
111	→P	
112	X⇄Y	print both parts of a complex number (may be R/S statements if desired)
113	PRTX	
114	X⇄Y	
115	PRTX	
116	GTO1	goto space and return subr
117	*LBL2	complex add subroutine
118	X⇄Y	
119	R↓	
120	+	
121	R↓	
122	+	
123	R↑	
124	RTN	
125	*LBL3	complex multiply subroutine
126	STOE	
127	R↓	
128	STOI	
129	R↓	
130	ENT↑	
131	R↑	
132	x	
133	R↑	
134	RCLI	
135	X⇄Y	
136	x	
137	LSTX	
138	R↓	
139	-	
140	R↑	
141	RCLE	
142	x	
143	R↑	
144	RCLI	
145	x	
146	+	
147	X⇄Y	
148	RTN	
149	*LBLc	POLAR/RECTANGULAR TOGGLE
150	CF1	indicate rectangular format
151	CLX	and place a zero in display
152	RTN	return to keyboard control
153	*LBLc	
154	SF1	indicate polar format and
155	EEX	place a one in the display
156	RTN	return to keyboard control
157	*LBLd	MATRIX INTERCHANGE
158	8	initialize index
159	STOI	
160	*LBL6	loop start
161	GSB4	recall corresponding matrix elements
162	DSZI	
163	GSB4	
164	ISZI	
165	GSB5	interchange and store corresponding matrix elements
166	DSZI	
167	R↓	
168	R↓	
169	GSB5	
170	DSZI	decrement index and test for loop exit
171	GTO6	
172	GTO1	goto space and return subr
173	*LBLD	CALCULATE MATRIX A TRANSPOSE
174	3	recall and scratchpad store a_{12}
175	GSB9	
176	STOA	
177	X⇄Y	
178	STOB	
179	5	recall a_{21} and store in a_{12} location
180	GSB9	
181	3	
182	GSB8	
183	RCLB	recall a_{12} from scratchpad and store in a_{21} location
184	RCLA	
185	5	
186	GSB8	
187	GTO1	goto space and return subr
188	*LBL9	store recall index subr
189	STOI	
190	R↓	
191	*LBL4	complex recall subroutine
192	P⇄S	
193	RCLi	
194	P⇄S	
195	RCLi	
196	RTN	
197	*LBL8	store storage index subr
198	STOI	
199	R↓	
200	*LBL5	complex storage subroutine
201	STOi	
202	X⇄Y	
203	P⇄S	
204	STOi	
205	P⇄S	
206	RTN	
207	*LBLE	CALCULATE MAT A COMPLEX CONJ
208	P⇄S	reverse the sign of the imaginary parts of the matrix elements
209	1	
210	CHS	
211	ST×7	
212	ST×5	
213	ST×3	
214	ST×1	
215	P⇄S	
216	*LBL1	space and return subroutine
217	SPC	
218	RTN	

LABELS

A	B	C	D	E
load A	load B	$A^{-1} \rightarrow A$	$A^{T} \rightarrow A$	$A^{*} \rightarrow A$
a print A	b print B	c polar/rect	d A⇄B	e calc $\lvert A \rvert$
0 complex print	1 space & return	2 complex add	3 complex multiply	4 complex recall
5 complex store	6 A⇄B subroutine	7 common print routine	8 sto index & cmplx store	9 store index & cmplx rcl

FLAGS

Flag	Use
0	
1	polar
2	used with $\lvert A \rvert$
3	

SET STATUS

FLAGS	ON	OFF	TRIG	DISP
0			users choice: DEG	FIX
1		■	GRAD	SCI
2		■	RAD	ENG
3				n ____

Part 5

ENGINEERING MATHEMATICS

PROGRAM 5-1 ELLIPTIC INTEGRALS AND FUNCTIONS.

Program Description and Equations Used

This program calculates complete elliptic integrals of the first kind and the following elliptic functions: elliptic sine (sn(u,k)), ellipic cosine (cn(u,k)), elliptic delta (dn(u,k)), and elliptic amplitude (am(u,k)).

The elliptic integral of the first kind is defined by Eq. (5-1.1), and the complete elliptic integral of the first kind is defined by Eq. (5-1.2), which can be evaluated using the infinite product shown in Eqs. (5-1.3) through (5-1.6). The product is terminated when k_m becomes smaller than 10^{-10}. Generally this condition is achieved after the 3rd term of the series, hence, the series converges rapidly. As the modulus, k, approaches 1, more iterations are required, e.g., K(.9) = 2.280549137 requires 4 iterations and K(.999) = 4.495596396 requires 5 iterations.

$$u(\phi,k) = \int_0^{\phi} (1 - k^2 \sin^2 x)^{-\frac{1}{2}} dx \qquad (5\text{-}1.1)$$

$$K(k) = u(\tfrac{\pi}{2}, k) \qquad (5\text{-}1.2)$$

$$K(k) = \frac{\pi}{2} \prod_{m=0}^{\infty} (1 + k_{m+1}) \qquad (5\text{-}1.3)$$

$$k_{m+1} = (1 - k_m')/(1 + k_m') \qquad (5\text{-}1.4)$$

$$k_m' = \sqrt{1 - k_m^{\ 2}} \qquad (5\text{-}1.5)$$

$$k_0 \equiv k \qquad (5\text{-}1.6)$$

The elliptic modulus, k, is commonly expressed three different ways, which leads to some degree of confusion. In the Abramowitz and Stegun tables of elliptic functions [1], the parameters m and θ are used where $m = k^2$, and $\theta = \sin^{-1} k$. The parameter θ is called the modular angle.

The elliptic sine is an elliptic function, and is defined in a somewhat reverse manner from the elliptic integral. Referring to Eq. 5-1.1, given the input $u(\phi,k)$, the limit of integration, ϕ must be found to satisfy the equality, then $sn(u,k) = \sin \phi$. Likewise, the elliptic cosine is defined; $cn(u,k) = \cos \phi$. Notice that when $k = 0$, the elliptic sine equals the trigonometric sine and likewise for the respective cosines.

The descending Landen transformation [12], [46] is used to calculate the elliptic sine. Starting with an initial value for $sn(u_r,k_r)$ as given by Eq. (5-1.7), Eq. (5-1.8) is recursively used to find $sn(u_0, k_0)$ which is the answer.

$$sn(u_{m+1}, k_{m+1}) = \sin\left(\frac{\pi u_0}{2K(k)}\right) \qquad (5\text{-}1.7)$$

$$sn(u_{r-1}, k_{r-1}) = \frac{(1 + k_r)\,sn(u_r,k_r)}{1 + k_r\,sn^2(u_r,k_r)} \qquad (5\text{-}1.8)$$

where

$$r = m+1,\ m,\ \ldots,\ 1$$

and k_r is obtained from storage, and was calculated from Eq. (5-1.4) during the complete elliptic integral calculation.

The descending Landen transformation is also the basis for Darlington's elliptic filter algorithms (Program 2-15).

The other elliptic functions are calculated from the elliptic sine as follows:

$$cn(u,k) = (1 - sn^2(u,k))^{\frac{1}{2}} \qquad (5\text{-}1.9)$$

$$dn(u,k) = (1 - k^2 \cdot sn^2(u,k))^{\frac{1}{2}} \qquad (5\text{-}1.10)$$

$$am(u,k) = \sin^{-1} sn(u,k) = \phi \qquad (5\text{-}1.11)$$

5-1

User Instructions

ELLIPTIC INTEGRALS AND FUNCTIONS				
complete elliptic integral, K(k)	elliptic sine sn(u,k), u↑k	elliptic cosine cn(u,k), u↑k	elliptic delta dn(u,k), u↑k	print ↑ / elliptic amplitude am(u,k), u↑k

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Select print/no-print option		[f] [E]	0 (R/S)
			[f] [E]	1 (print)
			[f] [E]	0 (R/S)
				⋮
3	For complete elliptic integral, K(k)	k	[A]	K(k)
4	For elliptic sine, sn(u,k)	u	[ENT↑]	
		k	[B]	sn(u,k)
5	For elliptic cosine, cn(u,k)	u	[ENT↑]	
		k	[C]	cn(u,k)
6	For elliptic delta, dn(u,k)	u	[ENT↑]	
		k	[D]	dn(u,k)
7	For elliptic amplitude, am(u,k)	u	[ENT↑]	
		k	[E]	am(u,k)

Example 5-1.1

Evaluate the following elliptic functions and compare with Abramowitz and Stegun [1] Tables 17.1 and 17.5.

$$K(k);\ k = \sqrt{0.9}$$

$$sn(3.09448898,\ \sin 88°)$$

HP-97 printout

```
             .9   √X
9.486832981-01   ***   calculate k
                GSBA
2.578092113+00   ***   K(k)


    3.09448898 ENT↑    load u
           88.   DEG
                 SIN   calculate k = sin 88°
9.993908270-01   ***   sin 88°
                GSBE
9.961546951-01   ***   sn(3.09448898, sin 88°)

                 DEG
                SIN⁻   calculate and print Ø = sin⁻¹sn(u,k)
8.500000000+01   ***
```

From Table 17.1 (p. 608 of [1]), K(m) for m = 0.9 is:

$$K(m) = 2.57809211334173$$

Rounded to ten significant figures, this figure agrees identically with the program output.

From Table 17.5 (p. 615 of [1]), the elliptic integral of the first kind for α = 88°, Ø = 85° is 3.09448898. The program output differs by 1 part in 8.5 x 10^9, which exceeds the precision of the input.

5-1 **Program Listing I**

Step	Key	Comment
001	*LBLA	COMPUTE COMPLETE ELLIPTIO INT
002	GSB2	calculate K(k)
003	GTO9	goto output routine
004	*LBL2	K(k) calculation subroutine
005	STO0	store k
006	Pi	calculate and store: $\frac{\pi}{2} \rightarrow R1$
007	2	
008	÷	
009	STO1	
010	EEX	$10 \rightarrow R8$
011	1	
012	STO8	
013	STOI	$10^{-10} \rightarrow R9$
014	EEX	
015	CHS	
016	1	
017	0	
018	STO9	
019	*LBL0	K(k) loop start
020	EEX	calculate and store: $k'_m = (1 - k_m^2)^{1/2}$
021	RCL0	
022	X²	
023	-	
024	√X	
025	STO7	
026	CHS	calculate and store: $k_{m+1} = \frac{1 - k'_m}{1 + k'_m}$
027	EEX	
028	+	
029	RCL7	
030	EEX	
031	+	
032	÷	
033	STO0	
034	STOi	store k_r for descending Landen transformation
035	ISZI	
036	EEX	form $\prod (1 + k_{m+1})$
037	+	
038	ST×1	
039	RCL9	test for loop exit
040	RCL0	
041	X>Y?	
042	GTO0	
043	RCL1	recall K(k)
044	RTN	return to main program
045	*LBLB	CALCULATE ELLIPTIO SINE
046	GSB3	calculate sn(u,k)
047	GTO9	goto output routine
048	*LBL3	sn(u,k) calculation subr
049	STO2	store k
050	R↓	recover and store u
051	STO3	
052	RCL2	calculate K(k)
053	GSB2	
054	DSZI	setup for sn(u,k) calc.
055	RAD	
056	RCL3	form and store initial sn value for descending Landen transformation: $sn(u_{m+1}, k_{m+1}) = sn\left\{\frac{\pi u_0}{2K(k)}\right\}$
057	Pi	
058	×	
059	RCL1	
060	ENT↑	
061	+	
062	÷	
063	SIN	
064	STO4	
065	*LBL1	transformation loop start
066	RCLi	recursively use descending Landen transformation to find $sn(u_0, k_0)$: $sn(u_{r-1}, k_{r-1}) = \frac{(1+k_r)sn(u_r, k_r)}{1 + sn^2(u_r, k_r)}$
067	EEX	
068	+	
069	RCL4	
070	×	
071	RCLi	
072	RCL4	
073	X²	
074	×	
075	EEX	
076	+	
077	÷	
078	STO4	
079	DSZI	test for loop exit
080	RCLI	
081	RCL8	
082	X≤Y?	
083	GTO1	
084	RCL4	recall sn(u, k)
085	RTN	return to main program

REGISTERS

0	1	2	3	4	5	6	7	8	9
k_i	K(k)	k_0	u_0	sn(u, k)			scratch	10	10^{-10}

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
k_1	k_2	k_3	k_4	k_5	k_6				

A	B	C	D	E	I

Program Listing II

```
086  *LBLC   CALCULATE ELLIPTIC COSINE
087   GSB3   calculate sn(u,k)
088   GTO6   convert to cn(u,k) & output
089  *LBLD   CALCULATE ELLIPTIC DELTA
090   GSB3   calculate sn(u,k)
091   RCL2   form k·sn(u,k) and convert
092      x   to dn(u,k) then output
093  *LBL6
094     X²   routine to calculate:
095    CHS
096    EEX          (1 - (·)²)^½
097      +
098     √X
099   GTO9   goto output routine
100  *LBLE   CALCULATE ELLIPTIC AMPLITUDE
101   GSB3   calculate sn(u,k)
102  SIN⁻¹   convert to am(u,k)
103  *LBL9   output subroutine
104    F0?
105   PRTX   print and space if
106    F0?   flag 0 is set
107    SPC
108    RTN   return to main program
109  *LBL7   R/S lockup routine
110    R/S
111   GTO7
112  *LBLe   PRINT - R/S  TOGGLE
113    F0?   jump if flag 0 is set
114   GTO8
115    SF0   set flag 0 and place a 1
116    EEX   in the display
117   GTO7   goto R/S lockup routine
118  *LBL8
119    CF0   clear flag 0 and place a
120    CLX   0 in the display
121    RTN   return control to keyboard
```

Flag 0 should be set (cleared) prior to magnetic card recording depending whether the user normally wants the program in the print (R/S) mode.

LABELS					FLAGS	SET STATUS		
A K(k)	B sn(u,k)	C cn(u,k)	D dn(u,k)	E am(u,k)	0 print	FLAGS	TRIG	DISP
a	b	c	d	e print toggle	1	ON OFF 0 ☐ ☐ 1 2 3	DEG GRAD RAD ■	FIX ■ SCI ENG n 9
0 K(k) loop	1 sn loop	2 K(k)	3 sn(u,k)	4	2			
5	6 $\sqrt{1-x^2}$	7 R/S lock	8 print toggle	9 print or R/S	3			

PROGRAM 5-2 BESSEL FUNCTIONS AND FM OR PHASE MODULATION SPECTRA.

Program Description and Equations Used

This program will calculate the magnitude of the spectral lines arising from a frequency of phase sine-wave modulation process. In addition, the power in the higher sidebands is calculated which can be used to help define the bandwidths necessary for a communication channel carrying frequency division multiplexed data with either frequency modulation (FM), or phase modulation (PM) on the individual subcarriers. Phase modulation is often used to transmit digital data with preconditioning such as Manchester biphase coding, or doublet modulation.

The spectra of both frequency modulated and phase modulated signals are the same when expressed as a function of the modulation index, m. The modulation index for the FM case is:

$$m_f = \frac{\text{peak carrier deviation from nominal frequency}}{\text{modulation frequency}}$$

Notice that the FM modulation index is modulation frequency dependent. The modulation index for the PM case is:

$$m_p = \begin{cases}\text{carrier phase shift in radians produced by the}\\ \text{modulating frequency.}\end{cases}$$

Also notice that the PM modulation index is modulation frequency independent.

The carrier and carrier sideband levels are described in terms of Bessel functions with the modulation index as the argument. The spacing of the sidebands is equal to the modulating frequency. For example, with a modulation index of 5 and a modulation frequency of 15 kHz, the FM or PM spectra is:

carrier amplitude	$J_0(5)$,
first sideband pair	$J_1(5)$,
second sideband pair	$J_2(5)$,
⋮	
n-th sideband pair	$J_n(5)$.

Figure 5-2.1 shows the above concept graphically.

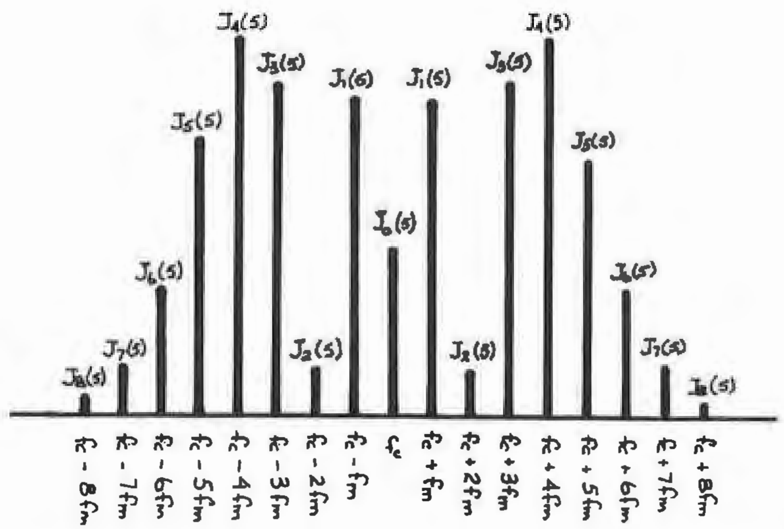

Figure 5-2.1 FM or PM modulation spectra.

A Bessel function identity allows the power remaining in the higher sidebands to be calculated. With FM or PM, all the sidebands carry modulation information in somewhat redundant form. If the higher order sidebands are removed by filtering, the modulation information can still be recovered, but the effective power will be reduced hence, the signal-to-noise ratio decreased; some distortion will also be introduced.

The Bessel function identity is:

$$J_0^2(m) + 2\sum_{i=1}^{\infty} J_i^2(m) = 1 \tag{5-2.1}$$

The summation is broken into 2 parts and the equation rearranged:

$$\sum_{i=n+1}^{\infty} J_i^2(m) = \tfrac{1}{2}(1 - J_0^2(m)) - \sum_{i=1}^{n} J_i^2(m) \tag{5-2.2}$$

Therefore, if the magnitudes of the first n sidebands are known, then the power in the higher sidebands may be calculated since power is proportional to magnitude squared.

When the modulating signal contains 2 sinewaves of different frequencies and amplitudes superposition does not hold, since the resulting

spectra is represented by the products of the Bessel functions of the individual spectra. Let m_1 be the modulation index for modulation frequency f_1, and likewise, m_2 for f_2, then the combined modulation spectral components will be as shown in Table 5-2.1.

Table 5-2.1 Spectra for combined modulation

Spectral Component	frequency of component	amplitude of component
Carrier	f_c	$J_0(m_1)\cdot J_0(m_2)$
Simple sidebands of	$f_c \pm f_1$	$J_1(m_1)\cdot J_0(m_2)$
	$f_c \pm f_2$	$J_0(m_1)\cdot J_1(m_2)$
	$f_c \pm 2f_1$	$J_2(m_1)\cdot J_0(m_2)$
	$f_c \pm 2f_2$	$J_0(m_1)\cdot J_2(m_2)$
	⋮	⋮
Intermodulation	$f_c \pm f_1 \pm f_2$	$J_1(m_1)\cdot J_1(m_2)$
	$f_c \pm f_1 \pm 2f_2$	$J_1(m_1)\cdot J_2(m_2)$
	$f_c \pm 2f_1 \pm f_2$	$J_2(m_1)\cdot J_1(m_2)$
	⋮	⋮

The Bessel function of the first kind is easily evaluated using the summation of an infinite series; however, for values of m larger than 10, computational difficulties arise because of small differences between big numbers, i.e., using Eq. (5-2.3), Table 5-2.2 shows the individual terms for $n = 0$ and $m = 20$.

$$J_n(m) = \left(\frac{m}{2}\right)^n \sum_{i=0}^{\infty} \frac{\left(-\frac{m^2}{4}\right)^i}{i!\cdot(i+n)!} = \left(\frac{m}{2}\right)^n \sum_{i=0}^{\infty} T_j \tag{5-2.3}$$

Table 5-2.2
Infinite series terms.

1.000000000	T_0
-100.0000000	T_1
2500.000000	
-27777.77778	
173611.1111	
-694444.4444	T_5
1929012.346	
-3936759.889	
6151187.327	
-7594058.428	
7594058.428	T_{10}
-6276081.347	
4358389.823	
-2578928.890	
1315780.046	
-584791.1313	T_{15}
228434.0357	
-79042.91893	
24395.96263	
-6757.884387	
1689.471097	T_{20}
-383.1000219	
79.15289702	
-14.96274047	
2.597697998	
-0.415631680	T_{25}
0.061483976	
-0.008434016	
0.001075767	
-0.000127915	
0.000014213	T_{30}
-0.000001479	
0.000000144	
-0.000000013	
0.000000001	

The computed $J_0(20)$ by this method is 0.166021646. Because the range of the numbers exceed 10^{10}, the least significant figures have been lost. Even though the summation was carried out until $T_i < 10^{-9}$, the answer is only accurate to 2 significant figures. The correct answer to $J_0(20)$ is 0.1670246646, which is computed by a slower, less direct method shown next.

Equation (5-2.4) is the recursion relationship for Bessel functions of the first kind.

$$J_n(m) = \frac{2}{m}(n-1)\cdot J_{n-1}(m) - J_{n-2}(m) \qquad (5\text{-}2.4)$$

All Bessel functions approach zero as the order becomes large. This characteristic can be used to compute Bessel functions. If $T_{n+2}(m) = 0$ and $T_{n+1}(m) = 10^{-9}$, the recursion relationship can be run backwards to arrive at a result that is proportional to $J_0(m)$. Abramowitz and Stegun [1] has the relations for the minimum starting index

and the constant of proportionality for $J_0(m)$, i.e., given

$$T_i(m) = \frac{2}{m}(i+1)\cdot T_{i+1}(m) - T_{i+2}(m) \qquad (5\text{-}2.5)$$

then, the minimum starting index is

$$i_{min} = 2.\text{INT}\left(\frac{6 + \max(n,z) + (9z/(z+2))}{2}\right) \qquad (5\text{-}2.6)$$

which for $n = 0$ may be reduced to

$$i_{min} = 2\cdot\text{INT}\left(\frac{z^2 + 17\cdot z + 12}{2(z+2)}\right) \qquad (5\text{-}2.7)$$

where

$$z = 3m/2 \qquad (5\text{-}2.8)$$

and "INT" means the integral part of the expression. The constant of proportionality is given by Eq. (5-2.9)

$$k = T_0(m) + 2\sum_{j=1}^{\frac{i_{min}}{2}} T_{2j}(m) \qquad (5\text{-}2.9)$$

The first two Bessel functions are then:

$$J_0(m) = \frac{T_0(m)}{k} \qquad (5\text{-}2.10)$$

$$J_1(m) = \frac{T_1(m)}{k} \qquad (5\text{-}2.11)$$

With $J_0(m)$ and $J_1(m)$ and the recursion relationship given by Eq. (5-2.4), all the higher order Bessel functions may be evaluated.

5-2 User Instructions

BESSEL FUNCTIONS AND FM OR PM MODULATION SPECTRA			Output Format: 0, 1, 2 ··· ; $J_0(m)$, $J_1(m)$, $J_2(m)$ ··· ; $(1-J_0^2)/2$, $10\log\sum_2^\infty J_i^2$, $10\log\sum_3^\infty J_i^2$ ···
load m & start	print ?		

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of program card			
2	Select print/no-print (R/S) option		B	0 (R/S)
			B	1 (print)
			B	0 (R/S)
				⋮
3	Load modulation index and start	m	A	0
				$J_0(m)$
				$(1-J_0^2)/2$
				1
				$J_1(m)$
	remaining power in higher sidebands in dB →			$10\log\sum_2^\infty J_i^2(m)$
				2
				$J_2(m)$
				$10\log\sum_3^\infty J_i^2(m)$
4	To stop analysis (print mode selected)		R/S	

Example 5-2.1

The 400 MHz carrier from a navigation satellite is phase modulated with a 400 Hz sinewave causing 60 degrees peak modulation. What is the modulation index, and what are the amplitudes of the PM sidebands?

The modulation index is the peak modulation expressed in radians:

$$m_p = 2\pi(60/360) = 1.0472 \text{ radians} \qquad (5\text{-}2.12)$$

HP-97 PRINTOUT FOR EXAMPLE 5-2.1

60.	D→R	
	GSBA	load modulation index and start
0.	***	carrier
0.744072	***	$J_o(m)$
0.223176	***	$(1 - J_o^2(m))/2$
1.	***	first sideband, $f_c \pm f_m$
0.455051	***	$J_1(m)$
-11.4	***	relative power in higher sidebands in dB
2.	***	second sideband, $f_c \pm 2f_m$
0.124972	***	$J_2(m)$
-26.4	***	
3.	***	
0.022322	***	$J_3(m)$
-44.0	***	
4.	***	
0.002964	***	$J_4(m)$
-63.5	***	
5.	***	
0.000313	***	$J_5(m)$
-84.5	***	

Notice that 99% of the power is contained in the carrier and the first two sidebands (-26.4 dB = 0.23% remaining power in higher sidebands).

Example 5-2.2

Calculate the sideband structure of a commercial FM station transmitting a 15 kHz signal with 75 kHz peak carrier deviation. The modulation index is:

$$m_f = 75000/15000 = 5 \quad (5\text{-}2.13)$$

HP-97 PRINTOUT FOR EXAMPLE 5-2.2

5. GSBA		load m_f & start
0.	***	carrier
-0.177597	***	$J_0(m)$
0.484230	***	$(1 - J_0^2(m))/2$
1.	***	first sidebands
-0.327579	***	$J_1(m)$
-1.1	***	power (dB) outside
2.	***	2nd sideband pair
0.046565	***	$J_2(m)$
-1.1	***	
3.	***	
0.364831	***	$J_3(m)$
-3.0	***	
4.	***	
0.391232	***	$J_4(m)$
-7.4	***	
5.	***	
0.261141	***	$J_5(m)$
-13.8	***	
6.	***	
0.131045	***	$J_6(m)$
-21.8	***	
7.	***	
0.053376	***	$J_7(m)$
-31.2	***	
8.	***	
0.018405	***	$J_8(m)$
-41.7	***	
9.	***	
0.005520	***	$J_9(m)$
-53.3	***	
10.	***	$J_{10}(m)$
0.001468	***	
-65.7	***	

Notice that one-half the power is contained in the first 3 sidebands and 99% of the power is contained in the first 6 sidebands.

The sideband structure for this example is shown in Fig. 5-2.1.

5-2

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD m AND START
002	STO0	store m
003	F0?	double space if flag 0 set
004	SPC	
005	F0?	
006	SPC	
007	1	calculate minimum starting index plus two
008	.	
009	5	
010	×	
011	ENT↑	
012	ENT↑	
013	ENT↑	
014	1	
015	7	
016	+	
017	×	
018	2	$i_{min} = 2 \cdot \text{int}\left\{\frac{Z^2+17Z+12}{2(Z+2)}\right\}$
019	÷	
020	6	
021	+	
022	X⇄Y	
023	2	
024	+	
025	÷	
026	INT	
027	ENT↑	
028	+	
029	2	
030	+	
031	STOI	
032	2	calculate and store 2/m
033	RCL0	
034	÷	
035	STOB	
036	CLX	initialize T_{i+2} and $\Sigma T_{2j}(m)$
037	STOE	
038	STO9	
039	EEX	initialize T_{i+1}
040	CHS	
041	9	
042	STOD	
043	*LBL0	calculate T_1 and T_0
044	GSB1	calculate and store ΣT_{2j}
045	ST+9	
046	CF2	execute recursion formula
047	GSB1	
048	F2?	test for loop exit
049	GTO0	
050	CLX	initialize i
051	STOI	
052	GSB7	print i
053	RCLE	calculate and print $J_0(m)$:
054	RCL9	
055	ENT↑	
056	+	
057	RCLE	$J_0(m) = \frac{T_0(m)}{k}$
058	−	
059	STO2	
060	÷	
061	STO1	
062	GSB6	
063	X²	calculate, store and print:
064	CHS	$\frac{1 - J_0^2}{2}$
065	EEX	
066	+	
067	2	
068	÷	
069	STO5	
070	GSB9	
071	ISZI	increment and print i
072	GSB7	
073	RCLD	calculate, store, and print $J_1(m)$:
074	CHS	
075	RCL2	
076	÷	$J_1(m) = \frac{T_1(m)}{k}$
077	STO2	
078	GSB6	
079	X²	calculate and print power in higher sidebands using Eq. (5-2.2)
080	CHS	
081	STO6	
082	RCL5	
083	+	
084	GSB8	

REGISTERS

0	1	2	3	4	5	6	7	8	9
m	$J_0(m), J_{n-1}(m)$	k, $J_1(m), J_n(m)$			$(1-J_0^2)/2$	$\sum_{i=1}^{n} J_i^2(m)$			$\Sigma T_{2j}(m)$
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
n	2/m		T_i, T_{i+1}	T_{i+1}, T_{i+2}	i, n

5-2 Program Listing II

```
085  *LBL2   loop to calc Bessel function
086   RCL1   J(n-2)
087    CHS
088   RCL2   J(n-1)
089   STO1
090   RCLB   2/m
091      x
092   RCLI   n-1        Jn(m) = (2/m)(n-1)J(n-1)(m) - J(n-2)(m)
093      x
094      +
095   STO2   Jn
096   ISZI   increment n
097   GSB7
098   RCL2   recall and print Jn(m)
099   GSB6
100     X²   calculate and print power
101   ST-6   in higher sidebands
102   RCL6
103   RCL5
104      +
105   GSB8
106   GTO2
107  *LBL1   Ti(m) recursion subroutine
108   DSZI
109    SF2
110   RCLE   T(i+2)
111    CHS
112   RCLI   i+1
113   RCLB   2/m
114      x
115   RCLD   T(i+1)     Ti(m) = (2/m)(i-1)T(i+1)(m) - T(i+2)(m)
116   STOE
117      x
118      +
119   STOD   Ti
120    RTN
121  *LBL6   print in dsp 6 subroutine
122   DSP6
123   GTO6
124  *LBL7   print index in dsp 0 subr
125   DSP0
126   RCLI
127  *LBL6
128    F0?   print and return if flag 0
129   PRTX   is set, otherwise stop
130    F0?
131    RTN
132    R/S   stop and await R/S command
133    RTN
134  *LBL8   calculate & prt 10·log subr
135   RCL5
136      ÷
137    LOG
138    EEX
139      1
140      x
141   DSP1   set display format
142    RND
143  *LBL9   print and space if flag 0
144    F0?   is set, otherwise stop
145   PRTX
146    F0?
147    SPC
148    F0?
149    RTN
150    R/S   stop and await R/S
151    RTN
152   GTO4   program block
153  *LBLE   PRINT-R/S TOGGLE
154    F0?   jump if flag 0 is set
155   GTO3
156    SF0   set flag 0 and place
157    EEX   a one in the display
158   GTO4   goto R/S lookup routine
159  *LBL3
160    CF0   clear flag 0 and place a
161    CLX   zero in the display
162  *LBL4   R/S lookup routine
163    R/S
164   GTO4
```

Note:

Flag 0 should be set or reset prior to magnetic card recording to cause the program to initially be in the print or R/S mode respectively as users desire.

LABELS					FLAGS	SET STATUS		
A load m and start	B R/S - print toggle	C	D	E	0 print	FLAGS	TRIG	DISP
a	b	c	d	e	1	ON OFF 0 □ □ 1 2 ■ 3	DEG GRAD RAD	FIX ■ SCI ENG n 0
0 J_0 & J_1 calc loop	1 J_0 & J_1 calc loop	2 output calc loop	3 R/S - print toggle	4 R/S lockup	2 loop exit			
5	6 dsp6, print	7 dsp0, prt I	8 10log, dsp1	9 print & space	3			

PROGRAM 5-3 CURVE FITTING BY THE CUBIC SPLINE METHOD.

Program Description and Equations Used

This program will fit a cubic spline interpolating curve through 2 to 9 equally spaced points [31]. The cubic spline represents the shape of the curve that would be generated if a clock spring were threaded through the data points. This technique is often used by draftsmen to draw a smooth curve through given points. The shape of such a curve looks natural, and is generally the shape one would attempt to draw by hand.

Let the ordinates, y_i, be given at $x_i = x_1 + (i-1)\cdot h$, where $i = 1,2, \ldots, n$, and h is the point spacing. Furthermore, let y(x) be the interpolating curve that is fitted to these points, and let y_i' and y_i'' represent the first and second derivatives of y(x) evaluated at $x = x_i$. y(x) may be represented piecewise where the function and its first and second derivatives are matched at the boundaries. The first and last segments of the interpolating curve may have their first and second derivatives specified by the user. The individual cubic interpolating polynomial $f_i(x)$ can be expressed in terms of the ordinates y_i and y_{i+1}, and either their first or second derivatives. Both forms will provide the same y(x), but the second derivative form requires simpler calculations.

Assume the third derivative, $y'''(x)$, is constant in each interval, h. This assumption implies that $y''(x)$ is linear in x, i.e.,

$$f_i''(x) = y_i''\left\{1 - \frac{x - x_i}{h}\right\} + y_{i+1}''\left\{\frac{x - x_i}{h}\right\} \qquad (5\text{-}3.1)$$

Equation (5-3.1) is integrated twice with respect to x, and the constants of integration chosen so the boundary conditions are met to the extent that $f_i(x_i) = y_i$ $(i = 1,2,\ldots,n-1)$, and $f_{i-1}(x_i) - y_i$ $(i = 2,3, \ldots,n)$. The results of this integration yield:

$$f_i(x) = y_i(1 - (x-x_i)/h) + y_{i+1}(x-x_i)/h$$
$$- (h^2/6)(y_i'')\left[1 - (x-x_i)/h - (1 - (x-x_i)/h)^3\right]$$
$$- (h^2/6)(y_{i+1}'')\left[(x-x_i)/h - ((x-x_i)/h)^3\right] \qquad (5\text{-}3.2)$$

Since the first and second derivatives of the function must also match at the boundaries, Eq. (5-3.2) is differentiated with respect to x and evaluated at x_i:

$$f_i'(x_i) = (y_{i+1} - y_i)/h - (h/6)(2y_i'' + y_{i-1}'') \qquad (5\text{-}3.3)$$

and

$$f_{i-1}'(x_i) = (y_i - y_{i-1})/h + (h/6)(y_{i-1}'' + 2y_i'') \qquad (5\text{-}3.4)$$

Equating Eqs. (5-3.3) and (5-3.4) implying boundary match yields:

$$h \cdot y_{i-1}'' = 4h \cdot y_i'' + h \cdot y_{i-1}'' = (6/h)(y_{i-1} - 2y_i + y_{i+1}) \qquad (5\text{-}3.5)$$

where

$$i = 2, 3, \ldots, n-1.$$

This equation set represents n-2 equations in n unknowns. If the starting and ending second derivatives are specified (y_1'' and y_n''), then the number of unknowns is reduced by 2, and a solution exists to the equation set. This equation set may be expressed in matrix notation:

$$\begin{bmatrix} 4 & 1 & 0 & 0 \cdots\cdots & 0 \\ 1 & 4 & 1 & 0 \cdots\cdots & 0 \\ 0 & 1 & 4 & 1 & 0 \cdots 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 \cdots\cdots 0 & 1 & 4 & 1 \\ 0 & 0 \cdots 0 & 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} y_2'' \\ y_3'' \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2}'' \\ y_{n-1}'' \end{bmatrix} = \begin{bmatrix} (6/h^2)(y_1 - 2y_2 + y_3) - y_1'' \\ (6/h^2)(y_2 - 2y_3 + y_4) \\ (6/h^2)(y_3 - 2y_4 + y_5) \\ \cdot \\ \cdot \\ \cdot \\ (6/h^2)(y_{n-2} - 2y_{n-1} + y_n) - y_n'' \end{bmatrix} \qquad (5\text{-}3.6)$$

Because of the tridiagonal characteristic of Eq. (5-3.6), a Gauss reduction is an effective method for finding the values of the various second derivatives. Let,

$$d_i = (6/h^2)(y_{i-1} - 2y_i + y_{i+1}) \qquad (5\text{-}3.7)$$

and select $y_1'' = y_n'' = 0$ (another common selection is $y_1'' = y_2''/2$ and $y_n'' = y_{n-1}''/2$). If a recursion relationship is defined thus:

$${}_i4 = 1/(4 - {}_{i-1}4) \text{ for } i = \ , 1, \ldots, n-1 \qquad (5\text{-}3.8)$$

i.e.,

$${}_04 = 1/4 = 0.25$$
$${}_14\ (1/3.75 = 0.2666^{-}$$
$${}_24 = 1/(4 - {}_14) = .267857143$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

then the Gauss reduced matrix becomes:

$$\begin{bmatrix} (1/{}_04) & 1 & 0 \cdots\cdots & 0 \\ 0 & (1/{}_14) & 1 \cdots\cdots & 0 \\ 0 & 0 & (1/{}_24) \cdots\cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 \cdots\cdots & 0 & 0 & (1/{}_{n-3}4) \end{bmatrix} \cdot \begin{bmatrix} y_2'' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1}'' \end{bmatrix} = \begin{bmatrix} d_2 \\ d_3 - {}_04 \cdot d_2 \\ d_4 - {}_14(d_3 - {}_04 \cdot d_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad (5\text{-}3.9)$$

Equation (5-3.9) is evaluated by the program as shown by the flowchart in Fig. 5-3.1.

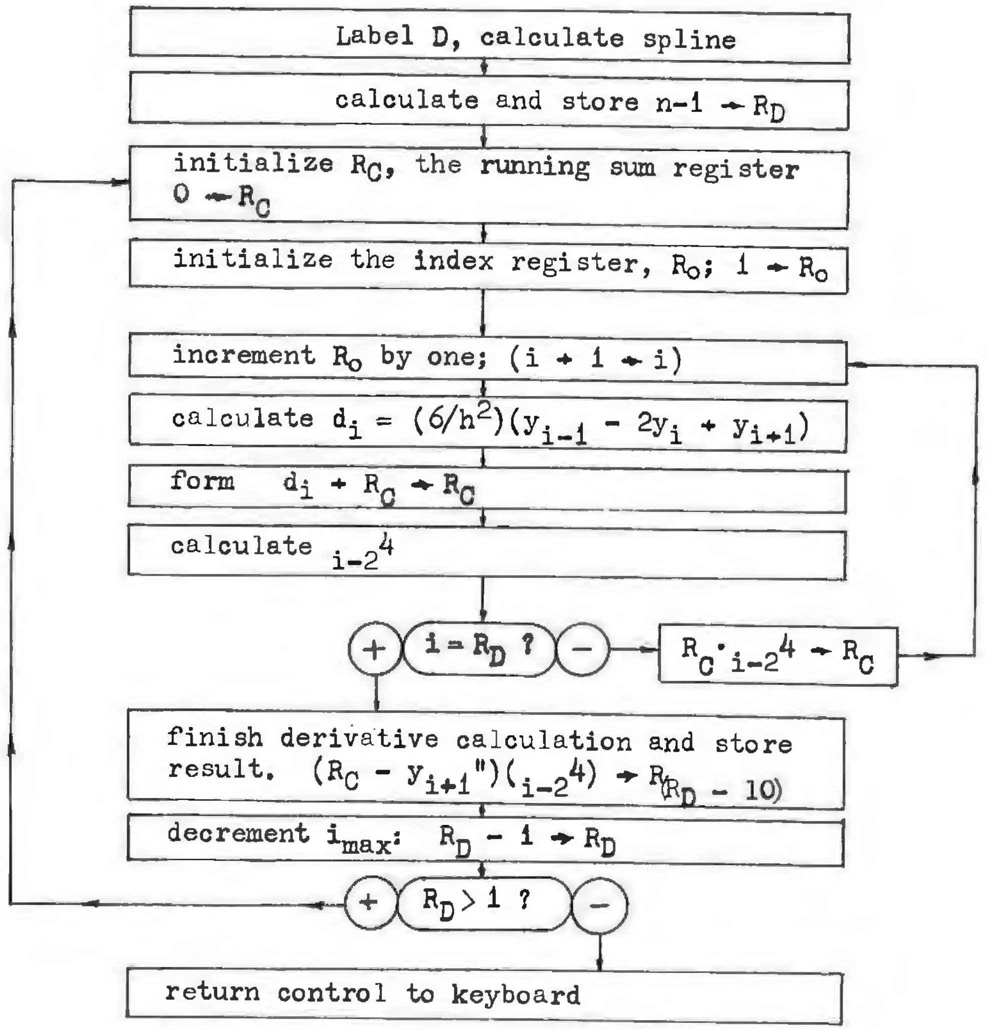

Figure 5-3.1 Flowchart of Gauss reduction algorithm.

5-3 **User Instructions**

CURVE FITTING BY THE CUBIC SPLINE METHOD				
Δx	start sweep			x_1
number of points	load h	load data, y_i	calculate spline	$x \to \hat{y}$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Load the number of data points	n	[A]	
3	Load h, the x interval	h	[B]	
4	Load y data	y_1	[C]	2
		y_2	[C]	3
		⋮		
		y_{n-1}	[C]	n
		y_n	[C]	
5	Calculate spline		[D]	
6	Load first x point	x_1	[f] [E]	
7	Execute single point interpolation	x	[E]	$\hat{y}$
	Step 7 may be used any number of times			
8	For linear sweep in x and corresponding interpolation of y:			
	a) Load sweep point spacing	Δx	[f] [A]	
	b) Start sweep		[f] [B]	x_1
				$\hat{y}_1$
				$x_1 + \Delta x$
				$\hat{y}$
				$x_1 + 2\Delta x$
				$\hat{y}$
				⋮
				$x_1 + (n-1)\Delta x$
				$\hat{y}$

Example 5-3.1

Fit a cubic spline interpolating curve to the data given in Table 5-3.1. Provide the output sweep with x increments of 0.1.

Table 5-3.1 Data for cubic spline interpolation.

x	1	2	3	4	5	6	7	8	9
y	0	5	9	7	4	3	5	8	9

The HP-97 printer output is shown on the next page, and the interpolated output is plated in Fig. 5-3.2. The bold points represent the given data.

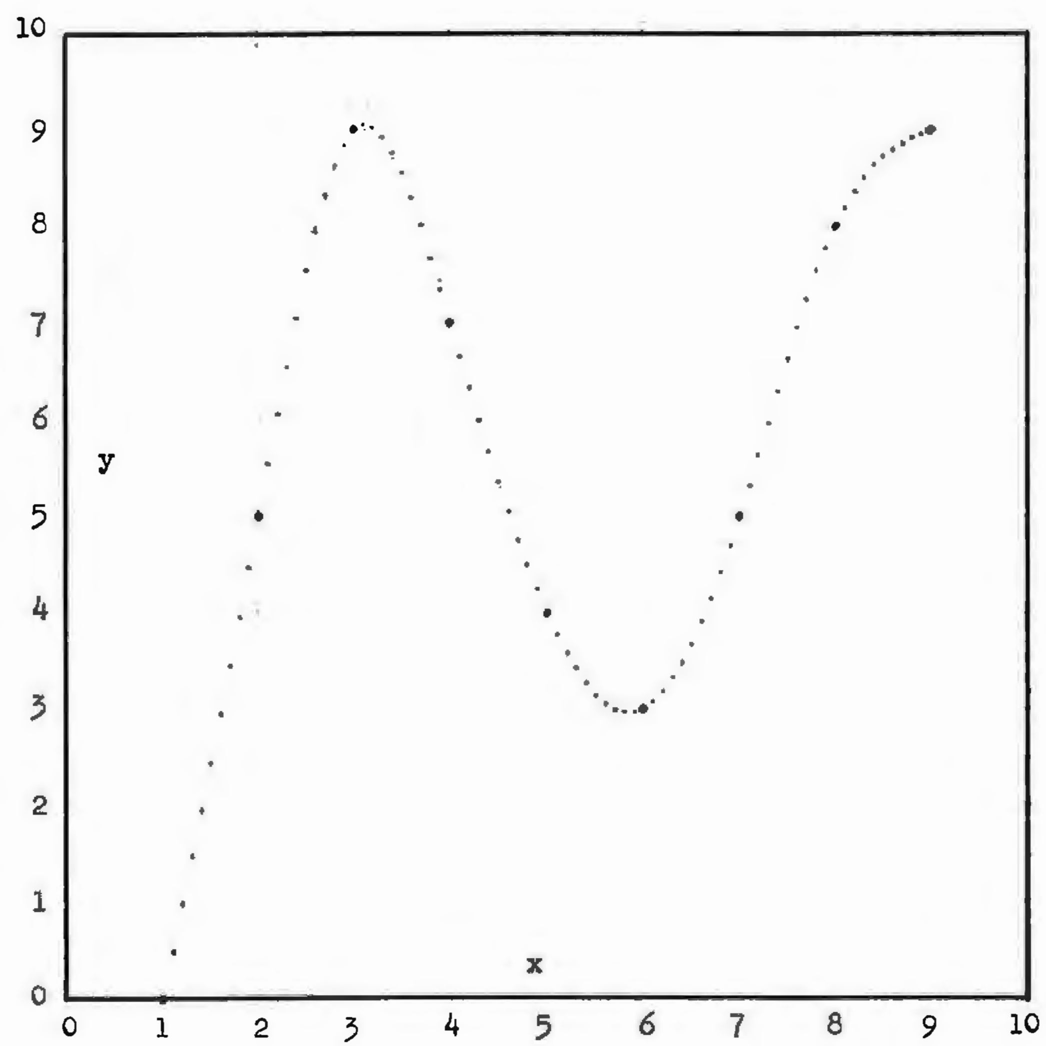

Figure 5-3.2 Cubic spline interpolation of given data.

HP-97 PRINTOUT FOR EXAMPLE 5-3.1

PROGRAM INPUT

9.000	GSBA	load number of data points
1.000	GSBB	load h, the x interval
		load y data points
0.000	GSBC	y_1
5.000	GSBC	y_2
9.000	GSBC	y_3
7.000	GSBC	y_4
4.000	GSBC	y_5
3.000	GSBC	y_6
5.000	GSBC	y_7
8.000	GSBC	y_8
9.000	GSBC	y_9
	GSBD	execute spline calculation
1.000	GSBe	load x_1, the first x point
.100	GSBa	load x interval for output sweep
	GSBb	start sweep

PROGRAM OUTPUT

1.000	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	
0.000	5.000	9.000	7.000	4.000	3.000	5.000	8.000	9.000	
1.100	2.100	3.100	4.100	5.100	6.100	7.100	8.100		x
0.486	5.530	9.059	6.656	3.783	3.073	5.315	8.196		y
1.200	2.200	3.200	4.200	5.200	6.200	7.200	8.200		
0.974	6.058	9.035	6.320	3.587	3.180	5.639	8.361		
1.300	2.300	3.300	4.300	5.300	6.300	7.300	8.300		
1.463	6.574	8.938	5.990	3.415	3.318	5.968	8.500		
1.400	2.400	3.400	4.400	5.400	6.400	7.400	8.400		
1.954	7.067	8.776	5.667	3.267	3.487	6.297	8.615		
1.500	2.500	3.500	4.500	5.500	6.500	7.500	8.500		
2.449	7.529	8.560	5.354	3.147	3.683	6.621	8.710		
1.600	2.600	3.600	4.600	5.600	6.600	7.600	8.600		
2.947	7.946	8.300	5.053	3.054	3.905	6.935	8.788		
1.700	2.700	3.700	4.700	5.700	6.700	7.700	8.700		
3.451	8.315	8.004	4.766	2.992	4.149	7.235	8.853		
1.800	2.800	3.800	4.800	5.800	6.800	7.800	8.800		
3.961	8.619	7.682	4.493	2.961	4.415	7.515	8.907		
1.900	2.900	3.900	4.900	5.900	6.900	7.900	8.900		
4.477	8.851	7.344	4.237	2.963	4.699	7.772	8.955		

Program Listing I

Key	Comments
*LBLA	LOAD # OF DATA POINTS
STOA	store number of data points
EEX	set y_n'' to zero
1	
+	
STOI	
CLX	
STOi	
EEX	initialize index register
STOI	
RTN	
*LBLB	LOAD h, THE x POINT
STOB	SEPARATION
RTN	
*LBLC	LOAD y DATA
STOi	store y data
ISZI	increment storage index
RCLI	recall index to display
RTN	return control to keyboard
*LBLD	CALCULATE SPLINE
RCLA	calculate and store n-1
EEX	
-	
STOD	
*LBL0	spline outer loop
CLX	initialize running sum
STOC	
EEX	initialize index register
STO0	
*LBL1	spline inner loop
EEX	increment and store index
ST+0	
RCL0	
EEX	
+	
STOI	
RCLi	calculate d_i
DSZI	
RCLi	
ENT↑	
+	
-	
DSZI	
RCLi	
+	$d_i = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$
6	
×	
RCLB	
X²	
÷	
RCLC	
-	$d_i - R_C \to R_C$
STOC	
RCL0	
2	
-	
X=0?	jump if i-2 is zero
GTO2	
STOI	initialize I
4	
*LBL3	calculate $({}_{i-2}4)^{-1}$
1/X	
CHS	
4	
+	
DSZI	
GTO3	
GTO4	
*LBL2	initialize $({}_{i-2}4)^{-1}$
4	
*LBL4	store ${}_{i-2}4$
1/X	
STOE	
RCL0	
RCLD	
X=Y?	test for loop exit
GTO2	
RCLC	
RCLE	
×	$R_0 \cdot {}_{i-2}4 - R_C$
STOC	
GTO1	goto inner loop start
*LBL2	finish derivative calc
RCL0	
1	calculate and store n+10 as storage index for derivative
1	
+	
STOI	
RCLC	
RCLi	calculate and store second derivative, y_i''
-	
RCLE	
×	
RCLD	
EEX	
1	$(R_C - y''_{i+1})({}_{i-2}4) \to R_{R_D+10}$
+	
STOI	
R↓	
STOi	
EEX	
RCLD	decrement i_{max}
EEX	
-	
STOD	
X>Y?	test for loop exit
GTO0	
SPC	
GTO8	

REGISTERS									
0 Δx for sweep & scratchpad	1 Y_1	2 Y_2	3 Y_3	4 Y_4	5 Y_5	6 Y_6	7 Y_7	8 Y_8	9 Y_9
S0 X_1	S1 current x for sweep	S2 Y_2''	S3 Y_3''	S4 Y_4''	S5 Y_5''	S6 Y_6''	S7 Y_7''	S8 Y_8''	S9 $Y_9'' = 0$
A n	B h	C scratchpad	D index	E scratchpad	I index				

Program Listing II

Key	Comments
*LBLe	LOAD FIRST x POINT (x_1 value)
P⇄S	store data
STO0	
P⇄S	
GTO8	
*LBLE	CALCULATE y ESTIMATE, x → ŷ
P⇄S	
RCL0	$x - x_1$
P⇄S	
-	
RCLB	
÷	$\frac{x - x_1}{h} \to R_C$
STOC	
INT	
EEX	$1 + INT\left(\frac{x - x_1}{h}\right) \to R_I$
+	
STOI	
RCLC	
-	$\frac{x_{i+1} - x}{h} \to R_D$
STOD	
RCLi	$\frac{x_{i+1} - x}{h} y_i$
×	
RCLC	
RCLI	
EEX	$\frac{x - x_i}{h} \to R_E$
-	
-	
STOE	
ISZI	
RCLi	
×	$(y_{i+1}) \frac{x - x_i}{h} + (y_i) \frac{x_{i+1} - x}{h} \to R_C$
+	
STOC	
RCLE	
RCLE	
3	$\frac{x - x_i}{h} - \left\{\frac{x - x_i}{h}\right\}^3$
Y^X	
-	
RCLI	
EEX	
1	
+	$(y''_{i+1}) \left\{ \frac{x - x_i}{h} - \left(\frac{x - x_i}{h}\right)^3 \right\}$
STOI	
R↓	
RCLi	
×	
RCLD	
RCLD	
3	$\frac{x_{i+1} - x}{h} - \left(\frac{x_{i+1} - x}{h}\right)^3$
Y^X	
-	
DSZI	
RCLi	$(y_i'') \left\{ \frac{x_{i+1} - x}{h} - \left(\frac{x_{i+1} - x}{h}\right)^3 \right\}$
×	
RCLI	
1	generate 0 if first interval otherwise generate 1 (frees up one register)
1	
-	
ENT↑	
X≠0?	
÷	
X=0?	
R↓	
×	compute running sum
+	
6	calculate $\frac{h^2}{6}$
÷	
RCLB	
X²	
×	
CHS	finish running sum calc
RCLC	
+	
PRTX	print y estimate (may be R/S statement)
SPC	
GTO8	goto R/S lockup
*LBLa	LOAD x FOR SWEEP
STO0	
GTO8	goto R/S lockup
*LBLb	START LINEAR SWEEP
SPC	
RCL0	
CHS	
P⇄S	
RCL0	initialize registers
+	
STO1	
P⇄S	
*LBL9	
RCL0	
P⇄S	increment x value
ST+1	
RCL1	
RCL0	
P⇄S	
RCLA	
EEX	calculate largest x value
-	
RCLB	
×	
+	
X⇄Y	
X>Y?	test for loop exit
GTO8	(may be R/S statement)
PRTX	print current x value
GSBE	calc and print y estimate
GTO9	
*LBL8	R/S lockup subroutine
RTN	
GTO8	

LABELS					FLAGS
A load nbr of data points	B load h, the x interval	C load y data	D calculate spline	E x → ŷ	0
a load Δx for sweep	b start sweep	c	d	e load first x point	1
0 loop destination	1 loop destination	2 initialize ${}_i4$ & scratch	3 calculate ${}_i4$	4 gauss reduction	2
5	6	7	8	9 loop destination	3

SET STATUS		
FLAGS	TRIG	DISP
ON OFF	USERS CHOICE	
0	DEG	FIX
1	GRAD	SCI
2	RAD	ENG
3		n____

PROGRAM 5-4 LEAST SQUARES CURVE-FIT TO AN EXPONENTIAL FUNCTION.

Program Description and Equations Used

Many processes both in electrical engineering and in physics have behavior that can be described by an exponential law, e.g., the voltage across a capacitor being charged through a series resistor asymptotically approaches the charging voltage in an exponential manner. When time constants are to be determined from oscilloscope photographs of these phenomena, only part of the entire waveform is available, and some error is introduced transferring the photograph data into numbers. If these errors are random, then a least squares fit can help remove them.

The equation form for the exponential function is given by:

$$x = a(1 - e^{-bt}) \tag{5-4.1}$$

Let d_i represent the difference between the measured point, x_i, and the exponential curve as shown by Fig. 5-4.1 and Eq. (5-4.2).

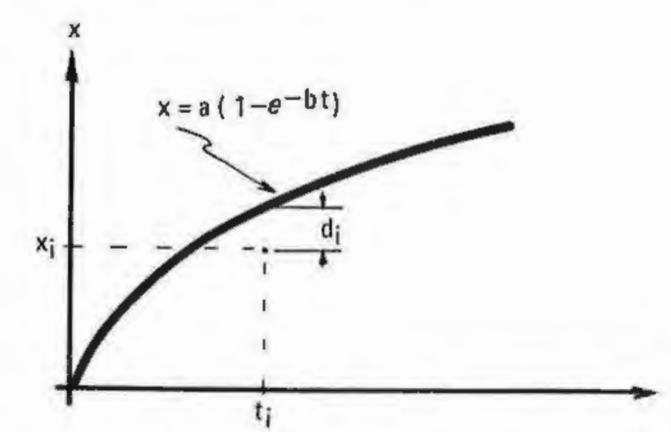

Figure 5-4.1 Exponential function.

$$d_i = x_i - a(1 - e^{-bt_i}) \tag{5-4.2}$$

The object of a least squares fit is to minimize the sum of the squares of the deviations as implied by Eq. (5-4.3).

$$S = \sum_i d_i^2 = \sum_i (x_i - a(1 - e^{-bt_i}))^2 \qquad (5\text{-}4.3)$$

The minimum can be found by setting the derivatives of Eq. (5-4.3) to zero, i.e.:

$$\frac{\partial S}{\partial a} = 0 \ , \ \frac{\partial S}{\partial b} = 0$$

or

$$\frac{\partial S}{\partial a} = -\ 2\sum_i \left\{x_i - a(1 - e^{-bt_i})\right\}\cdot(1 - e^{-bt_i}) = 0 \qquad (5\text{-}4.4)$$

$$\frac{\partial S}{\partial b} = -2a\sum_i \left\{x_i - a(1 - e^{-bt_i})\right\}\cdot(t_i\cdot e^{-bt_i}) = 0 \qquad (5\text{-}4.5)$$

Equations (5-4.4) and (5-4.5) represent 2 equations in 2 unknowns, a and b. Equation (5-4.4) is solved for a as shown in Eq. (5-4.6) and substituted into Eq. (5-4.5) to yield Eq. (5-4.7)

$$a = \frac{\sum_i x_i(1 - e^{-bt_i})}{\sum_i (1 - e^{-bt_i})^2} \qquad (5\text{-}4.6)$$

$$g(b) \triangleq \sum_i x_i\cdot t_i e^{-bt_i} \sum_i (1 - e^{-bt_i})^2 -$$

$$\sum_i x_i(1 - e^{-bt_i}) \sum_i t_i\cdot e^{-bt_i}(1 - e^{-bt_i}) = 0 \qquad (5\text{-}4.7)$$

To simplify things, the various sums in Eq. (5-4.7) are assigned numbers in the same respective order as they appear.

$$g(b) = \Sigma_1\ \Sigma_2 - \Sigma_3\ \Sigma_4 = 0 \qquad (5\text{-}4.8)$$

The object is to find b so g(b) = 0. Since Eq. (5-4.7) is nonlinear, an iterative solution is employed to find b. Wegstein's method [29] is used and is flowcharted in Fig. 5-4.2. This method is chosen because no derivatives are required and the convergence is very rapid.

Basically Wegstein's method is Esperti's method where one curve is a straight line (see Program 2-9 for Esperti's method). Equation (5-4.8) will have to be modified as Wegstein's method finds the solution to f(b) = b, therefore, let f(b) be as shown in Eq. (5-4.9)

$$f(b) = \frac{\Sigma_3\ \Sigma_4}{\Sigma_2} - \Sigma_1 + b \qquad (5.4\text{-}9)$$

The reason for this form is to try and avoid the small difference between big numbers problem. It is advisable to keep b between 0.1 and 10 for best accuracy. If the data is on a microsecond time scale, enter the time as though it were in seconds and denormalize b after it has been calculated. Likewise for millisecond data.

After b has been found by iteration, a is obtained by using Eq. (5-4.6), which can be expressed in terms of the numbered sums:

$$a = \frac{\Sigma_3}{\Sigma_2} \qquad (5.4\text{-}10)$$

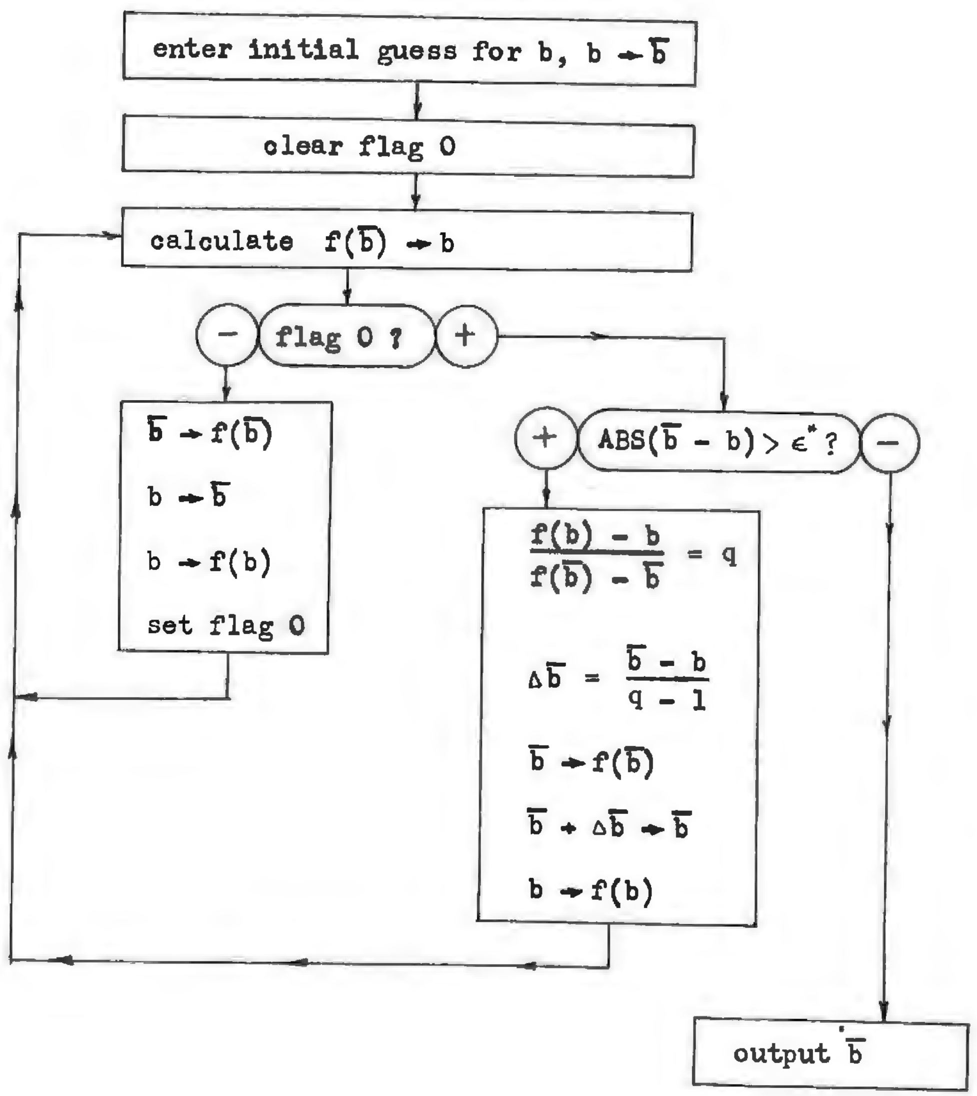

Figure 5-4.2 Flowchart for Wegstein's method.

* For this program, ε is chosen at $10^{-6} \cdot \bar{b}$

5-4 **User Instructions**

LEAST SQUARES FIT TO AN EXPONENTIAL FUNCTION				
compare input & least squares			print ?	clear input mode
$t_{start} \uparrow t_{stop} \uparrow \Delta t$	data entry	estimate b	start least squares fit	t → $\hat{x}$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Load both sides of magnetic card			
2	Select print or R/S option (toggle)		[f] [D] [f] [D] [f] [D]	0 (R/S) 1 (print) 0 (R/S) ⋮
3	Load t_{start}, t_{stop}, and t a) time of first data point b) time of last data point c) data point spacing	 t_{start} t_{stop} Δt	 [ENT] [ENT] [A]	
4	Load data (10 points maximum) load x at t_{start} load x at t_{start} + Δt ⋮ load last data point	 x_1 x_2 ⋮ x_n	[B] [B] [B] [B]	t_{start} t_{start} + Δt t_{start} + 2Δt ⋮ t_{stop} t_{stop} + Δt
5	Load estimate for b (0.1 ≤ b ≤ 10) To examine the currently stored value for b, key "C" without numeric entry. The input mode can be cleared with keys "f", "E".	$b_{estimate}$	[C]	
6	To clear input mode (used with step 5)		[f] [E]	
7	Start least squared fit		[D]	a b space
8	Optional; compare input data with least squares fit data		[f] [A]	t_{start} x_1 $\hat{x}_1$ ⋮
9	Calculate linear estimate for x given t	t	[E]	$\hat{x}$

Example 5-4.1

A constant voltage was suddenly connected to the field of a large dc traction motor, and an oscilloscope photograph taken of the current. The field time constant is needed to determine loop stability in the overall motor control loop. Table 5-4.1 shows the field current as read from the oscilloscope photo as a function of time.

Table 5-4.1 Motor field current vs. time.

time, seconds	0.1	0.2	0.3	0.4	0.5	0.6	0.7
current, amps	10	18	26	33	39	45	50

Assuming the field to be a simple series LR circuit, find the time constant and the asymptotic field current.

HP-97 PRINTOUT FOR EXAMPLE 5-4.1

```
 .100 ENT↑  load starting time
 .700 ENT↑  load stopping time
 .100 GSBA  load time increment

      GSBB  setup for data entry
10.000 GSBB load I(.1)
18.000 GSBB load I(.2)
26.000 GSBB load I(.3)
33.000 GSBB load I(.4)
39.000 GSBB load I(.5)
45.000 GSBB load I(.6)
50.000 GSBB load I(.7)

 1.000 GSBC load b estimate

      GSBD  start least sqrs fit

95.569 ***  a (asymptotic current)
 1.057 ***  b (1/τ, time constant)

τ = 1/1.057 = 0.9461 seconds
```

```
        GSBa  compare input & least squares
 0.100  ***   time
10.000  ***   I(t) input
 9.587  ***   I(t) from least sqs

 0.200  ***
18.000  ***
18.211  ***

 0.300  ***
26.000  ***
25.971  ***

 0.400  ***
33.000  ***
32.953  ***

 0.500  ***
39.000  ***
39.234  ***

 0.600  ***
45.000  ***
44.885  ***

 0.700  ***
50.000  ***
49.969  ***
```

5-4

Program Listing I

Step	Key	Comment
001	*LBLA	LOAD t_{start}, t_{stop}, Δt
002	STO0	store entries
003	R↓	
004	STOD	
005	R↓	
006	STOC	
007	GTOe	goto OF3 and R/S lockup subr
008	*LBLC	LOAD b ESTIMATE
009	F3?	if numeric input, store data
010	STOB	
011	RCLB	recall b estimate to display
012	GTO6	goto R/S lockup subroutine
013	*LBLD	START LEAST SQUARES CALC
014	CF0	indicate first time thru loop
015	*LBL9	outer loop start
016	CLX	initialize sums:
017	STO1	$0 \rightarrow \Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3 \rightarrow \Sigma_4$
018	STO2	
019	STO3	
020	STO4	
021	9	initialize index
022	STOI	
023	RCLC	initialize time register
024	RCL0	
025	-	
026	STO5	
027	*LBL0	summation loop start
028	ISZI	increment index
029	RCL0	increment time
030	ST+5	
031	RCLD	test for loop exit
032	RCL5	
033	X>Y?	
034	GTO3	
035	RCLB	e^{-bt_i}
036	×	
037	CHS	
038	e^x	
039	STO6	
040	CHS	$1 - e^{-bt_i}$
041	EEX	
042	+	
043	STO7	
044	X^2	$\Sigma_2 + (1-e^{-bt_i})^2 \rightarrow \Sigma_2$
045	ST+2	
046	RCLi	$\Sigma_1 + x_i \cdot t_i \cdot e^{-bt_i} \rightarrow \Sigma_1$
047	RCL5	
048	×	
049	RCL6	
050	×	
051	ST+1	
052	RCLi	$\Sigma_3 + x_i(1 - e^{-bt_i}) \rightarrow \Sigma_3$
053	RCL7	
054	×	
055	ST+3	
056	RCL5	$\Sigma_4 + t_i e^{-bt_i}(1 - e^{-bt_i}) \rightarrow \Sigma_4$
057	RCL6	
058	×	
059	RCL7	
060	×	
061	ST+4	
062	GTO0	goto summation loop start
063	*LBL3	start Wegstein solution
064	RCL3	$\frac{\Sigma_3 \cdot \Sigma_4}{\Sigma_2} - \Sigma_1$
065	RCL4	
066	×	
067	RCL2	
068	÷	
069	RCL1	
070	-	
071	F0?	jump if not first time through loop
072	GTO1	
073	RCLB	$\bar{b} \rightarrow f(\bar{b})$
074	STOE	
075	+	$b \rightarrow \bar{b} \rightarrow f(b)$
076	STO8	
077	STOB	
078	STO9	
079	SF0	not first time thru loop
080	GTO9	goto outer loop start
081	*LBL1	jump destination
082	RCLB	$f(\bar{b}) \rightarrow b$
083	+	
084	STO8	
085	RCLB	$ABS\left\{\frac{\bar{b}-b}{\bar{b}}\right\}$
086	-	
087	RCLB	
088	÷	
089	ABS	
090	EEX	test for loop exit
091	CHS	
092	6	
093	X>Y?	
094	GTO2	
095	RCL9	$q = \frac{f(b) - b}{f(\bar{b}) - \bar{b}}$
096	RCL8	
097	-	
098	RCLE	
099	RCLB	
100	-	
101	÷	
102	STOA	
103	RCLB	$\bar{b} \rightarrow f(\bar{b})$
104	STOE	
105	RCL8	$\Delta\bar{b} = \frac{\bar{b} - b}{q - 1}$
106	-	
107	RCLA	
108	EEX	
109	-	
110	÷	

REGISTERS

0	1	2	3	4	5	6	7	8	9
Δt	Σ_1	Σ_2	Σ_3	Σ_4	t_i	e^{-bt_i}	$1-e^{-bt_i}$	b	f(b)
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9

A	B	C	D	E	I
a, q	$\bar{b}$	t_{start}	t_{stop}	$f(\bar{b})$	index

Program Listing II

Step	Key	Comment
111	RCLB	$\bar{b} + \Delta\bar{b} \rightarrow \bar{b}$
112	+	
113	STOB	
114	RCL8	$b \rightarrow f(b)$
115	STO9	
116	GTO9	goto outer loop start
117	*LBL2	Wegstein output
118	F1?	space if print mode set
119	SPC	
120	RCL3	$a = \frac{\Sigma_3}{\Sigma_2}$
121	RCL2	
122	÷	
123	STOA	
124	GSB5	gosub print or R/S subr
125	RCLB	recall b
126	GTO4	goto print and space subr
127	*LBLc	COMPARE INPUT & LEAST SQRS
128	RCLC	setup time register and index register
129	RCL0	
130	-	
131	STO5	
132	9	
133	STOI	
134	*LBL7	loop start
135	ISZI	increment register index
136	RCL0	increment time index
137	S[illegible]	
138	RCL	test for loop exit
139	RCL5	
140	X>Y?	
141	GTO6	
142	GSB5	output time
143	RCLi	recall and output input
144	GSB5	
145	R↓	calculate and output least squares estimate
146	GSBE	
147	GTO7	goto loop start
148	*LBLE	LEAST SQUARES ESTIMATE
149	RCLB	calculate: $\hat{x} = a(1 - e^{-bt})$
150	×	
151	CHS	
152	e^x	
153	CHS	
154	1	
155	+	
156	RCLA	
157	×	
158	*LBL4	print and space subroutine
159	GSB5	gosub print or R/S subr
160	F1?	space if print mode set
161	SPC	
162	GTOe	goto CF3 and R/S lockup subr

Step	Key	Comment
163	*LBL5	print or R/S subroutine
164	F1?	print and return if flag 1 is set, otherwise
165	PRTX	
166	F1?	
167	RTN	
168	R/S	stop and await R/S command
169	RTN	
170	*LBLB	LOAD DATA
171	9	initialize register index
172	STOI	
173	*LBL8	data storage loop start
174	ISZI	increment register index
175	RCLI	calculate time for x(t)
176	EEX	
177	1	
178	-	
179	RCL0	
180	×	
181	RCLC	
182	+	
183	R/S	display time & await entry
184	*LBLB	data storage
185	STOi	store data
186	GTO8	goto loop start
187	*LBLe	CF3 and R/S lockup subr
188	CF3	clear flag 3
189	*LBL6	R/S lockup subroutine
190	RTN	
191	GTO6	
192	*LBLd	PRINT OR R/S TOGGLE
193	CF1	clear flag 1 for R/S mode
194	CLX	and place a zero in display
195	RTN	return control to keyboard
196	*LBLd	toggle continued
197	SF1	set flag 1 for print mode
198	EEX	and place a one in display
199	RTN	return control to keyboard

Note:
Flag one should be set or reset prior to magnetic card recording depending whether the user wishes the program to normally come up in print or R/S mode after the card read. Step 2 can be skipped in this instance.

LABELS					FLAGS
A load times	B load data	C load b estimate	D start least squares	E $t \rightarrow \hat{x}$	0 first time thru Wegstein
a output summary	b	c	d print toggle	e clear flag 3	1 print
0 form sums	1 Wegstein major loop	2 Wegstein output	3 Wegstein start	4 print	2
5 print or R/S	6 R/S lock	7 summary loop	8 data input loop	9 outer least squares loop	3

SET STATUS		
FLAGS	TRIG	DISP
ON OFF	USERS CHOICE	
0 ☐ ■	DEG	FIX
1 ☐ ☐	GRAD	SCI
2	RAD	ENG
3		n____

LIST OF ABBREVIATIONS

LIST OF ABBREVIATIONS

alternative or alternate	alt
amplifier	amp
approximately	approx
arithmetic	arith
attenuation	atten
bandpass	BP
bandstop	BS
bandwidth	BW
branch	br
Butterworth	Buttr
calculation or calculate	calc
capacitor	cap
Chebyshev	Cheb
circuit	ckt
clear	clr
coaxial	coax
coefficient	coef
complex	cmplx
conductance	cond
conjugate	conj
conversion	conv
co-ordinates	co-ords
decibel	dB
decibel ripple	dBR
denominator	den
denormalization	denorm
density	dens
destination	dest
diameter	diam
display	dsp
distance	dist
electrical	elect
elements	elts
enter	ent
Equation(s)	Eq(s).
equivalent	equiv
evaluation	eval
even part of (·)	Ev(·)
execute	exec
feedback	fdbk
Figure	Fig.
format	fmt
frequency	freq
function	fcn
go substitute	go sub
go to	gto
henry	h
highpass	HP
imaginary	imag
increment	incr
initialize	init
input/output	I/O
integral	int

label	lbl
level	lvl
linear	lin
loop	lp
lowpass	LP
matrix	mat
minimum	min
multiplication	mult
negative	neg
numerator	num
odd part of(·)	Odd(·)
order	ord
page	pg, p
parameters	params
peak	pk
polynomial	poly
preamplifier	preamp
primary	pri
print	prt
program	pgm
recall	rcl
rectangular	rect
reflection	refl
register(s)	reg(s)
required	reqd
resistance	resist
return	rtn
review	revu
root sum square	RSS
root mean square	RMS
secondary	sec
section	sect
solution	soln
space	spc
specification(s)	spec(s)
square	sq
starting frequency	f_{st}
stopping frequency	f_{sp}
store	sto
subroutine	subr
sweep	swp
temporary	temp
terminating, terminal, or termination	term
through	thru
toggle	tog
total	tot
transform	xfm
transformer	xfmr
transistor	xstr
transmitter	xmit
transmission	xmsn
trigonometric	trig

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