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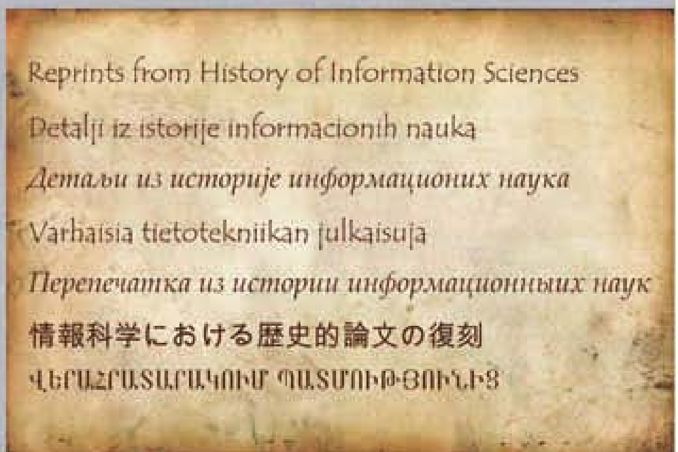
Reprints from the Early Days of Information Sciences

On the Contributions of P.S. Poreckij to Switching Theory



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Radomir S. Stanković & Jaakko Astola (eds.)

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Reprints from the Early Days of Information Sciences

TICSP Series

Solving General Tasks in Probability Theory by Using Mathematical Logic

a Lecture

by

P. S. Poreckij

2009

This publication has been written and edited by
Radomir S. Stanković and Jaakko T. Astola.

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Radomir S. Stanković.

Contents

Reprints from the History of Computing Science and Signal Processing

Historical studies about a scientific discipline is usually a sign of its maturity. When properly carried out, this kind of studies are more than enumeration of facts or giving credit to particular important researchers. It is more discovering and tracing the ways of thinking that have lead to important discoveries. In this respect, it is interesting and also important to recall publications where for the first time some important concepts, theories, methods, and algorithms have been introduced.

In every branch of science there are some important results published in national or local journals or other publications that have not been broadly distributed for different reasons, due to which they often remain unknown to the research community and therefore are rarely referenced. Sometimes, importance of such discoveries is overlooked or underestimated even by the inventors themselves. Such inventions are often re-discovered long after, but their initial sources remain almost forgotten, and mostly remain sporadically recalled and mentioned within quite limited circles of experts. This is especially often the case with publications in other languages than the English language which presently dominates the scientific world.

This series of publications is aimed at reprinting and, when appropriate, also translating some less known or almost forgotten, but important publications, where some concepts, methods or algorithms have been discussed for the first time or introduced independently of other related works.

R.S. Stanković, J. T. Astola

Solving General Tasks in Probability Theory by Using Mathematical Logic

by

P.S. Poreckij

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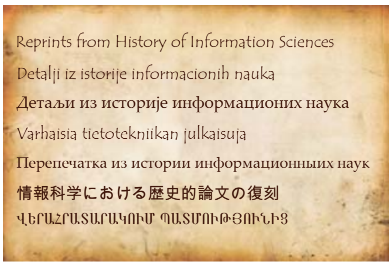

Radomir S. Stanković, Jaakko Astola

Solving General Tasks in Probability Theory by Using Mathematical Logic

Abstract

This issue of Reprints from the Early Days of Information Sciences presents an article by Platon Sergeevich Poreckij which is a record of his lecture delivered on October 25, 1886 at the 60th meeting of the Section for Physical and Mathematical Sciences of the Scientific Society of the Imperial University of Kazan, in Kazan, Russia. The article has been published by this Society as an official publication of the University of Kazan.

Poreckij, P.S., "Solving general tasks in probability theory by using mathematical logic", Izd-vo Kazan Univ, Kazan, Russia, 1887.

Solving General Tasks in Probability Theory by Using Mathematical Logic

In the year 1884, I published a treatise "On methods of solving logical equations", where it has been presented a complete theory of these equations.

Here, I intend to apply this theory to solving the following task in Probability Theory

Find the probability of a complex event which depends on some given simple events, by using the probabilities of all or few (arbitrarily selected) of these simple events as well as the probability of some other complex events under the assumption that the given events satisfy an arbitrary number of certain conditions.

It is obvious that this is the most general task regarding determination of the probability of events. As far as known to me, in Probability Theory there is not a method to find solution of this task in a general form. Therefore, a solution by exploiting Mathematical Logic cannot be considered as unnecessary [1].

§**1**. Before all, the question arises: *Is it possible to add the study of qualitative symbols (logic classes) to the study of quantitative (probabilistic) symbols*? The answer is: *It is possible.*

Essentially, the logic equality

$$f(a, b, c, d, \cdots) = \varphi(a, b, c, d, \cdots)$$

means that, in a sample space, all objects belonging to the class f, correspond to the objects in the class φ, and that the difference between the classes f and φ reduces to the different classification of the same objects. If so, the *number* of objects which belong to the classes f and φ must be equal, that is, for example

$$N[f(a, b, c, d, \ldots)] = N[\varphi(a, b, c, d \ldots)].$$

[1] The solution of this task provided by Boole in his treatise *An Investigation of the Laws of Thought* cannot be considered as the scientific one, since it is based on an arbitrary and purely empiric theory of logic equations, as well as since the idea of transition from logic equations to algebraic equations itself is not properly elaborated by Boole. In this manner, the main goal of the present article is to provide a scientific form to the deep, but vague and unrigorous idea of Boole on the applicability of Mathematical Logic in the Probability Theory.

This purely mathematical equality, follows directly from the initial logical equality. From there it is already easy to move further to the relations between probabilities. Denote by $N(1)$ the number of all objects in the sample space, and by $P(f)$ the relation $\frac{N(f)}{N(1)}$, i.e., the probability of the class f. Then, it is clear that

$$P[f(a,b,c,d,\ldots)] = P[\varphi(a,b,c,d,\cdots)].$$

Therefore, if two classes are logically equivalent, then their probabilities are mutually equal.

Due to that, the following general way for determining the probabilities can be disclosed: Find the logical relationship between events whose probability is required, and other events, whose probabilities are known, and then *make a transition* from logical equations between the events to algebraic equalities between their probabilities.

Now, we will consider construction of rules for such a transition from a logical equality to the corresponding algebraic equality.

§**2**. Let the logic symbols $a, b, c, \cdots$ denote simple events. In this case, logic negation (complement) of such symbols, i.e., $a_0, b_0, c_0, \cdots$, should denote correspondingly every event in the universe which is not a, every event except b, etc. Further, the logic sum as $a + b$, $a + b_0$ will denote the complex events, the first of which is occurrence of a or b, the second, occurrence of a or any other event except b, etc. Finally, logic product as ab, ab_0, etc., will denote a complex event which consists of in the first case - occurrence of a and b, in the second - occurrence of a and any other event except b, etc.

It is clear, for example, that the logic expression

$$a + b(c_0 + d_0) + b_0 d_0$$

denotes a complex event which occurs first in the case of occurrence of a, second in the case of occurrence of the conjunction of b and not c or b and not d, and finally third, in the case of occurrence of the conjunction of events not b and not d.

§**3**. From Probability Theory, is is known that the probability of non-occurrence of an event is equal to the unity (a certain event) minus the probability of its appearance.

If so, then

$$P(a_0) = 1 - P(a).$$

In the same way,

$$P(b_0) = 1 - P(b),$$

etc.

§**4**. Further, from Probability Theory it is known that if two events are mutually exclusive, then the probability of either occurring is equal to the sum of the their separate occurrences. Therefore, if the logic classes m and n are disjoint, i.e., do not contain common objects, (with $m \cdot n = 0$), then

$$P(m + n) = P(m) + P(n).$$

This rule can be applied to an arbitrary number of mutually exclusive events. For the application of this rule it is necessary to know how each logic sum

$$A + B + C + D + \cdots$$

can be converted into a disjunctive form, i.e., in the form

$$A + A_0B + A_0B_0C + A_0B_0C_0D + \cdots,$$

where A_0 is the negation of A, B_0 is the negation of B, etc., These two logic sums are logically equivalent, but differ in the fact that the above rule cannot be applied to the first of them, while being applicable to the second.

In this way, every complex event, that has the form of a sum, can be always expressed in such a manner that its probability is equal to the sum of the other, simpler, events. For example, the probability

$$P(A + B + C + D),$$

can be converted into the form

$$P(A + A_0B + A_0B_0C + A_0B_0C_0D),$$

which can be decomposed into the sum of probabilities

$$P(A) + P(A_0B) + P(A_0B_0C) + P(A_0B_0C_0D).$$

§**5**. Notice that it is known from Probability Theory, if two or more events are independent, then the probability of their joint occurrence is equal to the product of the probabilities of their separate occurrences. This means, if $a, b, c, \cdots$, are simple events not related by any logical relation, then

$$P(abc\cdots) = P(a)P(b)P(c)\cdots$$

§**6**. If so, then the probability that corresponds to the logic expression

$$A + A_0B + A_0B_0C + \cdots,$$

which does not satisfy any conditions, can be expressed as

$$P(A) + P(A_0)P(B) + P(A_0)P(B_0)P(C) + \cdots,$$

i.e., it is obtained from the probabilities of the simple events by the simple substitution of classes $A, B, C, \cdots$ and its logic complements by their probabilities.

From here we see that the absolute probability of each separate logic function

$$f(a, b, c, d, \ldots),$$

defined by a disjunctive form is

$$f[P(a), P(b), P(c), \ldots].$$

In the first of these expressions, f denotes the set of *logic* operations over the qualitative symbols $a, b, c, \ldots$; while in the second the same f denotes the set of *algebraic* operations over the quantitative symbols $P(a), P(b), P(c), \ldots$.
Example. If the probabilities of simple events x and y are $P(x) = p$ and $P(y) = q$, then the probability of the complex event $xy_0 + x_0y$, which is already in the disjunctive form, is $p(1-q) + (1-p)q$. The probability of the complex event $x + y$ which, when converted into the disjunctive form, is $x + x_0y$ or $y + y_0x$, can be expressed as $p + (1-p)q$, or $q + (1-q)p$.

This is the way to perform the transition from the expression of a given logic function to the expressions of its absolute probability.

§**7**. Now, it is known that for the transition from a logical equality $f = \varphi$ to the relationship between the probabilities of these classes, it is necessary to express both functions, f and φ, in the disjunctive form, and then replace on both sides of the sign of equality the qualitative symbols $a, b, c, \dots$ by the quantitative symbols $P(a), P(b), \dots$.

As an example, convert the logical equality

$$ab + cd = ac + bd,$$

into the relationship among the probabilities, by taking that $P(a) = p$, $P(b) = q$, $P(c) = r$, $P(d) = s$.

It is necessary to express both sides of the initial expression into the disjunctive form. We have,

$$\begin{aligned} ab + (ab)_0 cd &= ac + (ac)_0 bd \\ ab + (a_0 + b_0) cd &= ac + (a_0 + c_0) bd \\ ab + (a_0 + ab_0) cd &= ac + (a_0 + ac_0) bd \\ ab + a_0 cd + ab_0 cd &= ac + a_0 bd + ac_0 bd. \end{aligned}$$

In the last equality, both sides consists of terms which are mutually disjoint and, therefore, by performing the transition to relationships among the probabilities, we get

$$pq + (1-p)rs + p(1-q)rs = pr + (1-p)qs + p(1-r)qs.$$

§**8**. If we want, then we can always perform in the same way the transition from operations over logic equalities to relationships among probabilities. Equally, when solving tasks of determining probability of an event through probabilities of other events, it is natural to do as: *Find from the set of given logic conditions, the definition of the first event through other events, and then perform the transition to probabilities.* We will follow this approach.

§**9**. Until now, we were speaking about the absolute probabilities. Let us turn towards the conditional probabilities.

In Probability Theory the following fact has been proven. The probability of an event A, given the occurrence of some other event B, is equal to the joint probability of A and B, $P(AB)$ divided by the probability of A, $P(A)$, i.e., it is equal to the fraction $\frac{P(AB)}{P(A)}$.

Therefore, if $A = f(a, b, c, d \ldots)$, $B = \varphi(a, b, c, d, \ldots)$, then the required conditional probability can be obtained by replacing $a, b, c, d, \ldots$ in the expressions of the product of f and φ, represented in the disjunctive form, by their absolute probabilities and divide the result obtained by the expression for the function f represented in the disjunctive form, replacing at the same time the qualitative symbols by quantitative symbols. So, the required conditional probability will be

$$\frac{[f(a, b, c, \ldots)\varphi(a, b, c, \ldots)]}{[f(a, b, c, d, \ldots)]},$$

where the rectangular brackets denote the mentioned replacement (of variables).

As an example, assume $P(x) = p$, $P(y) = q$, $P(z) = r$, and find the probability of the event

$$xy_0 + x_0y,$$

i.e., if one of the events x and y, but no both, occur, then it will also occur the event

$$yz_0 + y_0z,$$

i.e., it will occur either y or z, but no both of them.

In this case,

$$f(x, y, z) = xy_0 + x_0y, \quad \varphi(x, y, z) = yz_0 + y_0z,$$
$$f(x, y, z)\varphi(x, y, z) = xy_0z + x_0yz_0.$$

It follows that the required probability is

$$\frac{[f\varphi]}{[f]} = \frac{p(1-q)r + (1-p)q(1-r)}{p(1-q) + q(1-p)}.$$

§**10**. Now, suppose that the given conditional probability of simple events $a, b, c, \ldots$, is determined such that fulfils a series of conditions

$$f'(a, b, c, \ldots) = \varphi'(a, b, c, \ldots), \quad f'' = \varphi'', f''' = \varphi''' \ldots,$$

and it is required to find the absolute probability of these simple events.

Notice before all that every logic equality

$$f(a, b, c, \ldots) = \varphi(a, b, c \ldots)$$

can be identically replaced by the equality

$$1 = f\varphi + f_0\varphi_0,$$

where 1 denotes the logic sample set (in this case, the set of all events in the question), f_0 and φ_0 are the logic complements of f and φ.

Besides that, it is known that all the given conditions are identical to the single condition

$$1 = (f'\varphi' + f_0'\varphi_0')(f''\varphi'' + f_0''\varphi_0'')(f'''\varphi''' + f_0'''\varphi_0''')\cdots,$$

which can be shortly written is the form

$$1 = M(a, b, c, d, \ldots).$$

In this equality, which equivalently replaces all the given conditions, the function M is called the logic sample set for the task considered, or the complete unity of the task.

So, fulfillment of all the initial conditions by the classes $a, b, c, \ldots$ is completely identical to the fulfilment of a single condition $1 = M(a, b, c, d, \ldots)$ constructed in the manner described above.

Suppose now that $p, q, r, \ldots$ denote probabilities of events $a, b, c, \ldots$, fulfilling the condition $1 = M$ and let $p', q', r', \ldots$ be the absolute probabilities of these events. Since the first of these probabilities denotes the probability that given the occurrence of M, the events $a, b, c, \ldots$ also occur, then for the determination of the absolute probabilities $p', q', r', \ldots$, we have

$$p = \frac{[aM]}{[M]}, \quad q = \frac{[bM]}{[M]}, \quad r = \frac{[cM]}{[M]}, \ldots,$$

where the classes $a, b, c, \ldots$ in rectangular brackets should be replaced by their absolute probabilities $p', q', r', \ldots$, which are determined by solving the system of obtained algebraic equations.

Consider an example. Suppose that when taking balls from a vase, we count just the cases when the balls inside are either white or blue (or both at the same time) and suppose that, under this condition, the probability to find a white ball is p and to find a blue one is q. Find the absolute probabilities p' and q'.

Construct first the condition under which the probabilities p and q have been determined. Let x denotes taking a white ball and y a blue one. If when calculating the probability we exclude the case that the balls inside are neither white nor blue, then we get the condition

$$x_0 y_0 = 0,$$

or, which is the same,

$$1 = xy + x_0 y + x y_0.$$

In this way, for the considered case

$$\begin{aligned} M(x,y) &= xy + x_0 y + x y_0, \\ xM(x,y) &= xy + x y_0 = x, \\ yM(x,y) &= xy + x_0 y = y. \end{aligned}$$

From there, we have

$$p = \frac{[Mx]_{x=p', y=q'}}{[M]_{x=p', y=q'}}, \quad q = \frac{[My]_{x=p', y=q'}}{[M]_{x=p', y=q'}},$$

or

$$p = \frac{p'}{p'q' + p'(1-q') + q'(1-p')}, \quad q = \frac{q'}{p'q' + p'(1-q') + q'(1-p')}.$$

By solving these two algebraic equations, we get

$$p' = \frac{p+q-1}{q}, \quad q' = \frac{p+q-1}{p}.$$

§**11**. Corresponding to that, as mentioned previously, for determining the probability of an event through the probabilities of some other events, we need to express the first event through these other events. This forces us to first say few words about defining a logic class (simple or complex) in terms of all or a few of these other classes.

Suppose that we want to determine a simple class a in terms of all other classes $b, c, d, \ldots$, related to a and also mutually related by the conditions (requirements)

$$f' = \varphi', \quad f'' = \varphi'', \quad f''' = \varphi''', \ldots.$$

All these conditions can be equivalently replaced by a condition

$$1 = M(a, b, c, d, \ldots).$$

On the other hand, this last equality can equivalently be replaced by the three equalities

$$\begin{aligned} a &= aM(1, b, c, d, \ldots) = aM(1), \\ a &= a + M(1, b, c, \ldots)M_0(0, b, c, \ldots) = aM(1)M_0(0), \\ 1 &= M(1, b, c, \ldots) + M(0, b, c, d, \ldots) = M(1) + M(0). \end{aligned}$$

Here, $M(1)$ is the result of substitution in $M(a, b, c, \ldots)$ the class a by 1, its logic complement a_0 by 0, $M(0)$ is the result of the substitution in $M(a, b, c, \ldots)$ the class a by 0 and its complement a_0 by 1; $M_0(0)$ is the logic complement of $M(0)$, or, which is the same, the result of substitution in the negation of M, i.e., in the function $M_0(a, b, c, \ldots)$, the class a by 0 and its complement a_0 by 1.

In latter three equalities, the fist shows that a is contained in $M(1)$, the second shows that a contains in itself $M_0(0)M(1)$. Due to this, these two equalities can be replaced by the inequalities

$$a < M(1), \quad a > M_0(0)M(1),$$

which should be understood in the sense: *a is not greater than $M(1)$ and no smaller than $M_0(0)M(1)$.*

Finally, the third equality $1 = M(1) + M(0)$, which depends on the classes $b, c, d \ldots$, but does not contain the class a, represents the condition, which, from the initial conditions, satisfy these both functions $M(1)$ and $M_0(0)M(1)$, in terms of which the a is defined.

In the case that these two functions are logically equivalent, i.e., when

$$M_0(0)M(1) = M(1),$$

two inequalities which determine a are

$$a > M(1), \quad a < M(1),$$

i.e., it is sufficient a single inequality

$$a = M(1).$$

If we want to determine a from this equality $1 = M(a, b, c, \ldots)$, not in terms of all, but of few classes $b, c, d, \ldots$, then all these classes have to be excluded from the equality $1 = M(a, b, c, d, \ldots)$. To do that, it is sufficient to replace in the equality $1 = M(a, b, c, \ldots)$ all classes that should be excluded and their logic complements by 1. Suppose that the result after excluding, would be $1 = M'$, where M' depends on a and few other classes. Then, it remains to determine a from the equality $1 = M'$ in the exactly the same way as it was determined from the equality $1 = M$.

This is the way to determine a simple class through all or few other simple classes based on an arbitrary number of given whatever logic conditions.

§**12**. Let us turn now to the determination of complex classes, i.e., functions.

It is easy to show that a logic function can be expressed in terms of simple classes (all or some of them) also when the latter do not satisfy any conditional equalities.

Actually, suppose that the given n simple classes $a, b, c, \ldots$ are mutually unrelated by any conditions, and let A be a complex class, where A is a specified particular function of these classes. In this case, let $A = w$, or, equivalently, $1 = Aw + A_0 w_0$. We can say that we have $n+1$ simple classes, $w, a, b, c, \ldots$, which satisfy the condition

$$1 = Aw + A_0 w_0 = M(w, a, b, c, \ldots).$$

From this condition, we can represent the simple class w (i.e., the function A) in terms of all or a few of these simple classes by the rules expressed above.

In this manner, consideration of an arbitrary logic function together with the independent simple classes converts the task without conditions into one with conditions.

If together with n mutually independent simple classes $a, b, c, \ldots$, we start considering m functions $U, V, W, \ldots$, then by introducing a series of notations

$$U = u, \quad V = v, \quad W = w, \ldots,$$

we get the task in terms of $n+m$ simple classes, $a, b, c, \ldots, u, v, w, \ldots$ which satisfy the condition

$$\begin{aligned} 1 &= (uU + u_0U_0)(vV + v_0V_0)(wW + w_0W_0)\cdots \\ &= M(a, b, c, \ldots, u, v, w, \ldots), \end{aligned}$$

from which, as shown above, we can determine logic definitions of all these classes $u, v, w, \ldots$ in terms of all or a few other classes, i.e., we can find any function $U, V, W, \ldots$ in terms of all or a few of those given simple classes and all or a few of these other functions.

Finally, if n simple classes $a, b, c, d, \ldots$ are mutually dependent and related though p conditions

$$A' = B', \quad A' = B'', \quad A''' = C''', \ldots,$$

where $A', B', A'', B'', \ldots$ are functions of $a, b, c, d, \ldots$, then in determining one of these m functions

$$U, V, W, \ldots$$

we will have the problem in terms of $n+m$ simple classes $a, b, c, d, \ldots, u, v, w, \ldots$ which are mutually related by $p+m$ conditions

$$A' = B', \quad A'' = B'', \quad \ldots, u = U, v = V, w = W, \ldots,$$

or, equivalently, the single condition

$$1 = (A'B' + A'_0B'_0)(A''B'' + A''_0B''_0)\cdots(uU + u_0U_0)(vV + v_0V_0)\cdots,$$

which can be rewritten as

$$1 = M(a, b, c, d, \ldots, u, v, w, \ldots).$$

From here, according to the previous considerations, we can find any of the functions $U, V, W, \ldots$ in terms of all or a few of these other functions, and also in terms of all or a few of the simple classes $a, b, c, d, \ldots$, where all the initial conditions will be taken into account.

§**13**. We have now all what is needed to solve the initial task of finding the probability of a function (a complex event) through the probabilities of all or a few other functions and simple classes, assuming that the latter are mutually related by an arbitrary number of whatever conditional equalities.

Suppose that by following the above method, we get the equality

$$1 = M(a, b, c, \ldots, u, v, w, \ldots),$$

from which are already excluded all classes and functions the probability of which should not be taken into account when determining the probability of the function U in terms of the probabilities of other classes $a, b, c, \ldots, v, w, \ldots$.

In this case, we get

$$u < M(1), \quad u > M_0(0)M(1),$$

where $M(1)$ and $M(0)$ are results of substitution in the function M the class u by 1 and 0 respectively, (and its logic complement by 0 and 1), where the other classes $a, b, c, \ldots, v, w, \ldots$ satisfy the relationship

$$1 = M(1) + M(0) = K.$$

It remains to determine the probability of u. Suppose that the probabilities of the classes $a, b, c, \ldots, v, w, \ldots$ are determined by taking into account all initial conditions for the given task. Consequently, they also satisfy the condition $1 = K$, and we denote them by $p, q, r, \ldots, \alpha, \beta, \ldots$. In this case, their absolute probabilities, which we denote by $p', q', r', \ldots, \alpha', \beta', \ldots$, should be determined from the requirement

$$p = \frac{[aK]}{[K]}, \quad q = \frac{[bK]}{[K]}, \quad \ldots \quad , \alpha = \frac{[vK]}{[K]}, \quad \beta = \frac{[wK]}{[K]}, \ldots,$$

where in the first part, after expressing the nominators and denominators in the disjunctive form, all qualitative symbols $a, b, c, \ldots, v, w, \ldots$ should be replaced by the qualitative symbols $p', q', r', \ldots, \alpha', \beta', \ldots$.

The values $p', q', r', \ldots, \alpha', \beta', \ldots$, determined in this way, will be substituted instead of $a, b, c, \ldots, v, w, \ldots$ in the first part of the inequality

$$u < M(1), \quad u > M_0(0)M(1),$$

which provides the absolute probability of the functions $M(1)$ and $M_0(0)M(1)$, i.e., it determines the absolute probability of the function u.

Equally, we need to know not absolute, but conditional probability of the function u, namely, the one in which are taken into account all the conditions given in this task, including correspondingly the condition $1 = K$. As shown before, such conditional probabilities of the functions $M(1)$ and $M_0(0)M(1)$, respectively, are

$$\frac{[M(1)K]}{[K]}, \quad \frac{[M_0(0)M(1)K]}{[K]},$$

where all the qualitative symbols $a, b, c, \ldots, v, w, \ldots$ should be replaced ny the corresponding absolute probabilities $p', q', r', \ldots, \alpha', \beta', \ldots$. Thus,

$$K = M(1) + M(0),$$

and therefore

$$\begin{aligned} M(1)K &= M(1)[M(1) + M(0)] = M(1), \\ M_0(0)M(1)K &= M_0(0)M(1)[M(1) + M(0)] = M_0(0)M(1). \end{aligned}$$

Correspondingly, the conditional probabilities of the functions $M(1)$ and $M_0(0)M(1)$ are

$$\frac{[M(1)]}{[K]}, \quad \text{and} \quad \frac{[M_0(0)M(1)]}{[K]}.$$

If so, then denoting by $P(u)$ the required conditional probability of the function u, we get

$$P(u) < \frac{[M(1)]}{[K]}, \quad P(u) > \frac{[M_0(0)M(1)]}{[K]}, \tag{1}$$

where all the qualitative symbols $a, b, c, \ldots, v, w, \ldots$ should be replaced by the symbols $p', q', r', \ldots, \alpha', \beta', \ldots$. After this substitution, these latter symbols should be replaced by their values expressed in terms of $p, q, r, \ldots \alpha, \beta, \ldots$ by the equalities

$$p = \frac{[aK]}{[K]}, \quad q = \frac{[bK]}{[k]}, \quad \ldots, \quad \alpha = \frac{[vK]}{[K]}, \quad \beta = \frac{[wK]}{[K]}, \ldots, \tag{2}$$

in which the substitution of symbols $a, b, s, \ldots, v, w, \ldots$ by the symbols $p, q, r, \ldots, \alpha, \beta, \ldots$ should be first performed. If in the formulaes (1) and (2), the qualitative symbols $a, b, c, \ldots, v, w, \ldots$ are replaced by the qualitative symbols $p', q', r', \ldots, \alpha'\beta', \ldots$, which should be afterwards excluded form (1) by using (2). Then it is clear that it is quite sufficient to view the qualitative symbols $a, b, c, \ldots, v, w, \ldots$ in (1) and (2) as some quantitative symbols and exclude them from (1) by using (2) and the rules of Algebra. In this way, the final form of the solution of the given task of determining $P(u)$ in terms of the conditional probabilities $p, q, r, \ldots, \alpha, \beta, \ldots$, becomes the following

By using the equalities

$$K = M(1) + M(0) = \frac{aK}{p} = \frac{bK}{b} = \cdots = \frac{vK}{\alpha} = \frac{wK}{\beta} = \cdots,$$

where, after expressing all the terms in the disjunctive form, the symbols $a, b, c, \ldots, v, w, \ldots$ *which are viewed as algebraic symbols, are excluded from the pair of inequalities*

$$P(u) < \frac{M(1)}{K}, \quad P(u) > \frac{M_0(0)M(1)}{K},$$

in which all the terms should be also expressed in the disjunctive form, and symbols $a, b, c, \ldots, v, w, \ldots$, *are also viewed as the quantitative symbols.*

This is the general method of solving the task formulated a the beginning of this paper. As we can see, in general, for the required probability $P(u)$ we get just the boundaries where it belongs, and just in the case when

$$M_0(0)M(1) = M(1),$$

we can get the exact solution for $P(u)$ as

$$P(u) = \frac{M(1)}{K}.$$

§**14**. Return to examples.

Example 1. Suppose that the probability that A or B (or both) will die in a given year is p. The probability that A or B (or both) will not die in the specified year is q. Find the probability that one of them will die while meanwhile the other will stay alive (e.g., either A will die while B will stay alive, or converse).

Let x be the event A will die, and y that B will die.

Given: $P(x+y) = p$, $P(x_0 + y_0) = q$. Find $P(xy_0 + x_0y)$.

We have three functions. Write

$$x + y = s, \quad x_0 + y_0 = t, \quad xy_0 + yx_0 = w.$$

The task can be viewed as that it contains five simple classes related though three conditions, or equivalently, by the following single condition

$$\begin{aligned} 1 &= [s(x+y) + s_0x_0y_0][t(x_0+y_0) + t_0xy] \\ &\quad \times[w(xy_0 + x_0y) + w_0(x_0y_0 + xy)] \\ &= stwxy_0 + stwx_0y + st_0w_0xy + s_0tw_0x_0y_0. \end{aligned}$$

From this equality, we should find the expressions for w in terms of s and t, and the classes x and y should be excluded (which can be achieved by the substitution $x = 1$, $y = 1$, $x_0 = 1$, $y_0 = 1$). The result of this excluding is

$$1 = M(s,t,w) = stw + st_0w_0 + s_0tw_0 = M(w),$$

from where

$$\begin{aligned} M(1) &= st, \quad M(0) = st_0 + s_0t, \quad M_0(0) = st + s_0t_0, \quad M_0(0)M(1) = st, \\ K &= M(1) + M(0) = s + s_0t, \quad Ks = s, \quad Kt = ts + ts_0 = t. \end{aligned}$$

Since in the given case, $M_0(0)M(1)$ is equal to $M(1)$, the two inequalities which determine the function w reduces to a single equality

$$w = M(1) = st.$$

Indeed, the multiplication of $s = x + y$ by $t = x_0 + y_0$ is $w = xy_0 + x_0y$.

Therefore, the required probability $P(w)$ is determined by the equality

$$P(w) = \frac{M(1)}{K} = \frac{st}{s + s_0t},$$

and after excluding from it the symbols s and t, since they are viewed to be quantitative, by the equality

$$K = s + s_0t = \frac{s}{p} = \frac{t}{q}.$$

From these equalities, we have

$$\begin{aligned} p &= \frac{s}{K}, \quad q = \frac{t}{K}, \quad p+q = \frac{s+t}{K}, \\ p+q-1 &= \frac{s+t-K}{K} = \frac{s+t-(s+(1-s)t)}{K} = \frac{t-t+ts}{K} = \frac{ts}{K}. \end{aligned}$$

Consequently, we finally have

$$P(w) = p + q + 1.$$

To check these observations, we notice the following. If $P(x+y) = p$, then $P[(x+y)_0] = P(x_0 y_0) = 1 - p$. In the same way, if $P(x_0 + y_0) = q$, then $P(xy) = 1 - q$.

It follows,

$$P(xy + x_0 y_0) = P(xy) + P(x_0 y_0) = 2 - p - q,$$

and then

$$P(xy_0 + x_0 y) = P[(xy + x_0 y_0)_0] = 1 - [2 - p - q] = p + q - 1,$$

which is the result that completely agrees with the one found above.

Example 2. Suppose that the probability that the speaker A tells the truth is p, and the probability that the speaker B is telling the truth is q, and the probability that their statements differs is r. Find the probability that if their statements agree, then we get the truth.

Suppose that the class of cases when speakers A and B tell the truth are x and y, respectively. Given:

$$P(x) = p, \quad P(y) = q, \quad P(xy_0 + x_0 y) = r.$$

Find the relationship

$$\frac{P(xy)}{P(xy + x_0 y_0)} = \frac{P(xy)}{1 - r}.$$

It is obvious that it is sufficient to find just $P(xy)$ in terms of p, q, and r.

Let

$$xy_0 + x_0 y = s, \quad xy = w.$$

These two conditions are equivalent to the single equality

$$1 = ws_0xy + w_0(sx_0y + s_0x_0y_0 + sxy_0).$$

This is the condition which should be satisfied in this particular task involving four simple classes x, y, s, w. It is required to find w in terms of the other three classes. We have

$$\begin{aligned}
1 &= ws_0xy + w_0(sx_0y + s_0x_0y_0 + sxy_0) = M(w), \\
M(1) &= s_0xy, \quad M(0) = s(x_0y + xy_0) + s_0x_0y_0, \\
M_0(0) &= s(xy + x_0y_0) + s_0(x + y), \quad M_0(0)M(1) = s_0xy = M(1), \\
K &= M(1) + M(0) = s_0xy + s_0x_0y_0 + sx_0y + sxy_0.
\end{aligned}$$

Since $M_0(0)M(1) = M(1) = s_0xy$, then instead of two inequalities, w is determined by a single equality

$$w = s_0xy.$$

Besides that,

$$Kx = s_0xy + sxy_0, \quad Ky = s_0yx + sx_0y, \quad Ks = sx_0y + sxy_0.$$

By viewing x, y, and s, as quantitative symbols, we should exclude them from the formulae

$$P(w) = \frac{s_0xy}{K}$$

by using the relationships

$$\begin{aligned}
\frac{xys_0 + xy_0s}{p} &= \frac{xys_0 + x_0ys}{q} = \frac{x_0ys + xy_0s}{r} = K \\
&= s_0xy + s_0x_0y_0 + sx_0y + sxy_0.
\end{aligned}$$

We have

$$\begin{aligned}
r &= \frac{sx_0y}{K} + \frac{sxy_0}{K}, \\
q &= \frac{s_0xy}{K} + \frac{sx_0y}{K}, \\
p &= \frac{s_0xy}{K} + \frac{sxy_0}{K} = \frac{s_0xy}{K} + \left(r - \frac{sx_0y}{K}\right) = \frac{s_0xy}{K} + r + \frac{s_0xy}{K} - q.
\end{aligned}$$

It follows

$$\frac{s_0 xy}{K} = \frac{p+q-r}{2}.$$

And finally,

$$\begin{aligned} P(w) &= \frac{p+q-r}{2}, \\ \frac{P(xy)}{P(xy+x_0y_0)} &= \frac{p+q-r}{2(1-r)}. \end{aligned}$$

Example 3. From the observation of an epidemic in some settlement, let p be the probability that a house suffered from fiver, q from cholera, and r is the probability that the house did not suffer from either of these diseases, providing sufficient sanitary conditions.

Find the probability that some particular house from this settlement does not fulfill the sanitary conditions.

Let x be the suffering from fiver, y from cholera, and z denotes that a house does not fulfill the sanitary conditions. Given:

$$P(x) = p, \quad P(y) = q, \quad P(x_0y_0z_0) = r.$$

Find $P(z)$.

Let

$$x_0y_0z_0 = w.$$

The condition, which is satisfied by the given task of four simple classes x, y, z, w is

$$1 = wx_0y_0z_0 + w_0(x+y+z) = F(z).$$

From here, find z in terms of x, y, w.

We have

$$\begin{aligned} F(1) &= w_0, \quad F(0) = wx_0y_0 + w_0(x+y), \\ F_0(0) &= w(x+y) + w_0x_0y_0, \quad F_0(0)F(1) = w_0x_0y_0. \end{aligned}$$

Consequently,

$$z < w_0, \quad z > w_0x_0y_0.$$

Besides that,

$$\begin{aligned} K &= F(1)+F(0) = w_0 + wx_0y_0 + w_0(x+y) = w_0 + wx_0y_0, \\ Kx &= xw_0, \quad Ky = yw_0, \quad Kw = wx_0y_0. \end{aligned}$$

The symbols w, x, y which are viewed as quantitative, should be excluded from the inequalities

$$P(z) < \frac{w_0}{K}, \quad P(z) > \frac{w_0x_0y_0}{K}$$

by using the relationships

$$\frac{xw_0}{p} = \frac{yw_0}{q} = \frac{wx_0y_0}{r} = K = w_0 + wx_0y_0.$$

We have

$$\begin{aligned} w_0 &= \frac{wx_0y_0}{r} - wx_0y_0 = \frac{wx_0y_0(1-r)}{r} = K(1-r), \\ \frac{w_0}{K} &= 1-r, \\ p+r &= \frac{xw_0 + wx_0y_0}{K}, \\ 1-p-r &= \frac{K - xw_0 - wx_0y_0}{K} = \frac{w_0 - xw_0}{K} = \frac{x_0w_0}{K}, \\ q+r &= \frac{yw_0 + wx_0y_0}{K}, \\ 1-q-r &= \frac{K - yw_0 - wx_0y_0}{K} = \frac{w_0 - yw_0}{K} = \frac{y_0w_0}{K}, \\ (1-p-r)(1-q-r) &= \frac{w_0^2x_0y_0}{K^2}, \\ \frac{(1-p-r)(1-q-r)}{1-r} &= \frac{w_0^2x_0y_0}{K^2} \cdot \frac{K}{w_0} = \frac{w_0x_0y_0}{K}. \end{aligned}$$

And then finally

$$P(z) < 1-r, \quad P(z) > \frac{(1-p-r)(1-q-r)}{1-r}.$$

Example 4. Suppose that regarding the balls in a vase, it is know that each ball is either big or not blue. Suppose that while taking the balls, we pay attention just to the cases when the balls inside are either white, or big, or blue. Suppose that under these conditions, the probability that the ball is white and big is p. Find the probability that the ball is either white but not big, or not white, but either big or blue.

Let x denote taking the white ball, y the big ball, and z the blue ball. The first two initial conditions of this task are

$$x = x(y + z_0),$$
$$1 = x + y + z.$$

Given is the probability $P(xy) = p$.

Find the probability $P(xy_0 + x_0(y + z))$.

Let

$$xy = u, \quad xy_0 + x_0(y + z) = v.$$

It is possible to say that the given task consists of 5 simple classes x, y, z, u, v which satisfy the above given four conditions. These conditions can be combined in the following single condition

$$\begin{aligned} 1 &= [x_0 + y + z_0][x + y + z][uxy + u_0x_0 + u_0y_0] \\ &\quad \times [vxy_0 + vx_0y + vx_0z + v_0xy + v_0x_0y_0z_0] \\ &= uv_0xy + u_0vx_0y + u_0vxz_0 + u_0vxy_0z_0. \end{aligned}$$

From the given task, it should be determined v in terms of u. The other classes x, y, z should be excluded which can be done by the substitution

$$x = y = z = x_0 = y_0 = z_0 = 1.$$

After their elimination, we get

$$1 = uv_0 + u_0v = F(v).$$

From there we have

$$F(1) = u_0, \quad F(0) = u, \quad F_0(0) = u_0, \quad F_0(0)F(1) = u_0.$$

It follows that in the case considered, v is determined by the equality

$$v = u_0.$$

Further, we have

$$K = F(1) + F(0) = u_0 + u = 1.$$

Correspondingly, the condition $1 = K$, which is satisfied by the function u, reduces to the identity $1 = 1$, which is identical to the absence of any condition. Therefore, we finally have

$$P(v) = P(u_0) = 1 - p.$$

Addendum

On the Numerization of Logic Equations in General

Above (§1) it was shown that to each logic equality

$$f(a, b, c, \ldots) = \varphi(a, b, c, \ldots) \tag{1}$$

corresponds the quantitative equality

$$N[f(a, b, c, \ldots)] = N[\varphi(a, b, c, \ldots)], \tag{2}$$

expressing equality of the number of objects contained in the classes f and φ.

Dividing both sides of this equality by $N(1)$, which denotes the number of objects in the sample space, another numerical equality is obtained

$$P[f(a, b, c, \ldots)] = P[\varphi(a, b, c, \ldots)], \tag{3}$$

expressing the equality of the probabilities of logic classes f and φ.

For short, the transition from the equality (1) to the equality (3) will be called the *probabilization* of the logic equation (1), while the transition from the equality (1) to the equality (2) will be called the *numerization* of the logic equality (1).

In the previous considerations, we were interested in the *direct* probabilization of logic equalities, for the transition from (1) directly to the e quality (3) without the intermediate equality (2). Doing that, for determining features of the symbol P, it was necessary to exploit several facts from Probability Theory.

If we will construct rules for the transition from the equality (1) to the equality (2), where for determining features of the symbol N we will already exploit facts from Probability Theory, then taking into account the simple relationship between the equalities (2) and (3), from these rules we will also obtain a new method for defining some features of the symbol P.

It is also needed to notice that the equality (2) can have some meaning not only as an intermediate rule between (1) and (3), but by itself, so that it could find some applications in other areas of knowledge, as for example in Statistics.

Return to the construction of rules for numerization of logic equalities.

For the numerization of a logic equality, it is sufficient to numerize each of its parts separately and then to mutually equate these results. In this manner, the numerization of logic equalities reduces to the numerization of separate logic functions.

Determination of the number of objects contained in every logic class a, i.e., determining the number $N(a)$ can be performed by their enumeration. Equivalently, given the relationship among some of the symbols $N(a)$, $N(b)$, $N(a+b)$, $N(ab)$, etc., we can determine the values of some of these symbols in terms of the values of other symbols.

Establishing various forms of relationships between different symbols N represents the subject of study of the Theory of Numerization.

First, we will find the relationship between two symbols $N[f_0(a,b,c,\ldots)]$ and $N[f(a,b,c,\ldots)]$, where f_0 is the logic complement of f.

From the logic identity

$$f(a,b,c,\ldots) + f_0(a,b,c,\ldots) = 1,$$

we have

$$N[f(a,b,c,\ldots) + f_0(a,b,c,\ldots)] = N(1).$$

Since the product $f \cdot f_0$ is equal to zero, then all objects of the function f differ from the objects of f_0, and then

$$N[f + f_0] = N(f) + N(f_0)$$

and correspondingly

$$N(f) + N(f_0) = N(1),$$

from where

$$N[f_0(a,b,c,\ldots)] = N(1) - N[f(a,b,c,\ldots)].$$

This is actually the required relationship. If we divide both sides by $N(1)$, we will get the relationship

$$P[f_0(a,b,c,\ldots)] = 1 - P[f(a,b,c,\ldots)],$$

i.e., one of the basic facts from Probability Theory.

Let us find the expressions for the symbol $N(a+b)$.

If a and b are disjoint, i.e., if $ab=0$, then is is clear that

$$N(a+b)=N(a)+N(b).$$

Suppose that a and b are related by the conjunction, i.e., that ab is different from zero. From the logic identity,

$$a=ab+ab_0,$$

where in the first part both terms are disjoint, we get

$$N(a)=N(ab)+N(ab_0).$$

In the same way, from the identity

$$b=ab+a_0b,$$

where again the first two terms are disjoint, we find

$$N(b)=N(ab)+N(a_0b).$$

By adding the expressions for $N(a)$ and $N(b)$, we will have

$$N(a)+N(b)=2N(ab)+N(ab_0)+N(a_0b).$$

On the other hand, the sum of the above expressions for a and b provides (from the general law of logic $m+m=m$) the logical equality

$$a+b=ab+ab_0+a_0b,$$

in the first part of which all three terms are mutually disjoint. Therefore,

$$N(a+b)=N(ab)+N(ab_0)+N(a_0b).$$

By comparing this expressions with the expression previously derived, shows that in general

$$N(a+b)=N(a)+N(b)-N(ab),$$

from where for the particular case when $ab=0$ and correspondingly $N(ab)=N(0)=0$, we get, the same as before,

$$N(a+b)=N(a)+N(b).$$

Further, it is easy to see, that in general (from the facts proven above, and also the law $mm = m$)

$$\begin{aligned}
N(a+b+c) &= N[(a+b)+c] = N(a+b) + N(c) - N[(a+b)c] \\
&= N(a) + N(b) - N(ab) + N(c) - N[ac+bc] \\
&= N(a) + N(b) + N(c) - N(ab) - [N(ac) + N(bc) - N(abc)] \\
&= [N(a) + N(b) + N(c)] - [N(ab) + N(ac) + N(bc)] + N(abc).
\end{aligned}$$

In the same way we found

$$\begin{aligned}
N(a+b+c+d) &= [N(a) + N(b) + N(c) + N(d)] \\
&\quad -[N(ab) + N(ac) + N(ad) + N(bc) + N(bd) + N(cd)] \\
&\quad +[N(abc) + N(abd) + N(bcd)] - N(abcd).
\end{aligned}$$

The laws of constructing such formulaes are obvious. In particular, when all the involved classes are mutually disjoint, we find

$$N\Sigma a^{(i)} = \Sigma N(a^{(i)}),$$

from where, after dividing by $N(1)$, we get the relationship

$$P(a' + a'' + a''' + \ldots) = P(a') + P(a'') + P(a''') + \cdots,$$

that is another truth from Probability Theory, to which we referred above.

It is possible to find another expression for the symbol $N\Sigma a^{(i)}$.

Since in logic there is the identity

$$a' + a'' = a' + a'_0 a'',$$

where the terms in the first part are mutually disjoint, then

$$N(a' + a'') = N(a') + N(a'_0 a'').$$

Further, by knowing that

$$a' + a'' + a''' = a' + a'_0 a'' + a'_0 a''_0 a''',$$

where again all the terms in the first part are disjoint, we find

$$N(a' + a'' + a''') = N(a') + N(a'_0 a'') + N(a'_0 a''_0 a'''_0).$$

In the same way, we have in general

$$N(a' + a'' + a''' + \cdots) = N(a') + N(a'_0 a'') + N(a'_0 a''_0 a''') + \cdots.$$

The third approach to determine the symbol $N\Sigma a^{(i)}$ consists of the decomposition of the sum $\Sigma a^{(i)}$ into elements (which are always mutually disjoint). Thus, if there is such a decomposition

$$\Sigma a^{(i)} = s' + s'' + s''' + \cdots,$$

then, it is clear that

$$N\Sigma a^{(i)} = N\Sigma s^{(i)}.$$

Finally, the forth approach to determine the same symbol is the following. Since the negation of the sum $a'+a''+a'''+\cdots$ is the product $a'_0 a''_0 a'''_0 \cdots$, then it is clear that

$$N(a' + a'' + a''' + \cdots) = N(1) - N(a'_0 a''_0 a'''_0 \cdots).$$

Let us return to the determination of the symbol N of the product of logic classes.

It was proven above that

$$N(a + b) = N(a) + N(b) - N(ab),$$

and therefore

$$N(ab) = N(a) + N(b) - N(a + b).$$

It is easy to see that

$$N(ab) = N(1) - N[(ab)_0] = N(1) - N(a_0 + b_0) \qquad (E)$$

I will not consider generalizations of these formulaes. Instead of that, let us pay attention to the following. The formulae before the last one shows that knowing the symbols $N(a)$ and $N(b)$, we still cannot determine the value of the symbol $N(ab)$. Also, is is easy to determine the area within which the value for this symbol should be contained, namely, $N(ab)$ is not smaller than 0 and not greater than the smaller of the values for the symbols $N(a)$ and $N(b)$.

We will prove that smaller of these boundaries can be formulated more precisely. Namely, it can be proven that $N(ab)$ is not smaller than

$$N(a) + N(b) - N(1).$$

Actually, from the equation (E), it follows that

$$\begin{aligned} N(ab) &= N(1) - N(a_0 + b_0) = N(1) - [N(a_0) + N(b_0) - N(a_0b_0)] \\ &= N(1) - [N(1) - N(a) + N(1) - N(b) - N(a_0b_0)] \\ &= N(a) + N(b) - N(1) + N(a_0b_0)]. \end{aligned}$$

This is a new expression for the symbol $N(ab)$, from where we can see that obviously $N(ab)$ is not smaller than

$$N(a) + N(b) - N(1).$$

Therefore, for the case of the product of three terms, we have

$$\begin{aligned} N(a'a''a''') &= N[(a'a'')a'''] = N(a'a'') + N(a''') - N(1) + N((a'_0 + a''_0)a''') \\ &= N(a') + N(a'') - N(1) + N(a'_0a''_0) + N(a''') \\ &\qquad -N(1) + N((a'_0 + a''_0)a'''_0) \\ &= N(a') + N(a'') + N(a''') - 2N(1) \\ &\qquad +[N(a'_0a''_0) + N((a'_0 + a''_0)a'''_0)]. \end{aligned}$$

Since each of the symbols N is not smaller than zero, it follows that $N(a'a''a''')$ is not smaller than

$$N(a') + N(a'') + N(a''') - 2N(1).$$

By such considerations it can be proven that in general

$$N(a'a''a''' \ldots a^{(m)}) \quad \text{not smaller than} \quad \Sigma N(a) - (m-1)N(1).$$

This is the lower bound of the value for the symbol N of the product of classes. Regarding the upper bound, it is clear that the value of this symbol is not greater than the value of the smallest of the symbols $N(a'), N(a''), \ldots, N(a^{(m)})$.

This is essentially all what I know about the rules of numerization.

To conclude, I would like to say the following. Above, we obtained from the rules of numerization two basic theorems from Probability Theory. Equally, we can obtain other theorems from this theory from the rules of numerization just by using the hypothesis about the uniform distribution of objects in each class over the entire sample space. For example, under the conditions in his hypothesis, we can say that $N(ab)$ is the same part of $N(a)$ as $N(b)$ is a part of $N(1)$, i.e., we can write the proportion

$$N(ab) : N(a) = N(b) : N(1),$$

from where

$$N(ab) = \frac{N(a)N(b)}{N(1)},$$

and correspondingly, after division by $N(1)$

$$P(ab) = \frac{N(a)}{N(1)} \cdot \frac{N(b)}{N(1)} = P(a)P(b).$$

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РѢШЕНІЕ ОБЩЕЙ ЗАДАЧИ ТЕОРІИ ВѢРОЯТНОСТЕЙ ПРИ ПОМОЩИ МАТЕМАТИЧЕСКОЙ ЛОГИКИ.

СООБЩЕНІЕ

П. С. Порѣцкаго,

читанное 25 октября 1886 г. въ 60-мъ засѣданіи секціи физико-математическихъ наукъ Общества Естествоиспытателей при Императорскомъ Казанскомъ Университетѣ.

Въ 1884 году я публиковалъ сочиненіе „О способахъ рѣшенія логическихъ равенствъ", гдѣ изложена полная теорія этихъ равенствъ.

Здѣсь я предполагаю примѣнить эту теорію къ рѣшенію слѣдующей задачи Теоріи Вѣроятностей: опредѣлить вѣроятность сложнаго событія, зависящаго отъ данныхъ простыхъ событій, помощію вѣроятностей всѣхъ или нѣсколькихъ (произвольно избранныхъ) изъ этихъ простыхъ событій, а также вѣроятностей нѣкоторыхъ другихъ сложныхъ событій, предполагая, что данныя событія подчинены произвольному числу какихъ-бы то ни было условій.

Очевидно, это есть самая общая задача относительно опредѣленія вѣроятностей событій. Сколько мнѣ извѣстно, въ Теоріи Вѣроятностей нѣтъ способа рѣшенія этой задачи въ общемъ видѣ. А потому рѣшеніе ея помощію Математич. Логики не должно представляться излишнимъ (1).

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(1) Рѣшеніе этой задачи, данное Булемъ въ его сочиненіи An investigation of the laws of thought, нельзя считать научнымъ, какъ потому, что оно основано на произвольной и чисто-эмпирической теоріи ло-

приложеніе ученія о качественныхъ символахъ (логическихъ классахъ) къ ученію о символахъ количественныхъ (вѣроятностяхъ)? Отвѣчаемъ: возможно.

Въ самомъ дѣлѣ, логическое равенство

$$f(a,b,c,d,...)=\varphi(a,b,c,d,...)$$

означаетъ, что, въ предѣлахъ нѣкотораго міра рѣчи, всѣ предметы, относящіеся къ классу f, вполнѣ тождественны съ предметами класса φ, и что все отличіе между классами f и φ заключается въ различной классификаціи однихъ и тѣхъ же предметовъ. Если такъ, то *число* предметовъ, содержащихся въ классахъ f и φ, должно быть одно и то-же, т. е. напр.

$$N[f(a,b,c,d...)]=N[\varphi(a,b,c,d...)].$$

Вотъ чисто-математическое равенство, прямо вытекающее изъ исходнаго логическаго. Отсюда уже легко перейти и къ отношенію между вѣроятностями. Если означимъ черезъ $N(1)$ число всѣхъ предметовъ міра рѣчи и назовемъ отношеніе $\frac{N(f)}{N(1)}$, т. е. вѣроятность класса f, символомъ $P(f)$, то понятно, что

$$P[f(a,b,c,d...)]=P[\varphi(a,b,c,d,...)].$$

Итакъ, если два класса логически равнозначны, то ихъ вѣроятности равны между собою.

гическихъ равенствъ, такъ и потому, что самая идея о переходѣ отъ логическихъ равенствъ къ алгебраическимъ разработана у Буля неудачно. Такимъ образомъ, главная цѣль настоящей статьи—дать научную форму глубокой, но смутной и бездоказательной, идеѣ Буля о примѣнимости Математич. Логики къ Теоріи Вѣроятностей.

Отсюда открывается слѣдующій общій путь для опредѣленія вѣроятностей: найти логическую связь между событіемъ, котораго вѣроятность ищется, и другими событіями, вѣроятности которыхъ даны, а затѣмъ сдѣлать *переходъ* отъ логическаго равенства между событіями къ алгебраическому равенству между ихъ вѣроятностями.

Построеніемъ правилъ для такого перехода отъ логическаго равенства къ соотвѣтственному алгебраическому намъ и предстоитъ теперь заняться.

§ 2. Пусть логическіе символы $a,b,c,\ldots$ означаютъ простыя событія. Въ такомъ случаѣ, логическія отрицанія тѣхъ-же символовъ, т. е. a_0, b_0, c_0,... должны означать соотвѣтственно: всякое, въ предѣлахъ міра рѣчи, событіе, только не a; всякое событіе, кромѣ b, и т. д. Затѣмъ, логическія суммы въ родѣ $a+b$, $a+b_0$ и т. д. должны означать сложныя событія, состоящія: первое—въ наступленіи или a, или b; второе—въ наступленіи или a, или всякаго событія, кромѣ b, и т. д. Наконецъ, логическія произведенія въ родѣ ab, ab_0 и т. д. должны означать сложныя событія, состоящія: первое—въ совпаденіи событій a и b, второе—въ совпаденіи событія a съ какимъ угодно событіемъ, кромѣ b, и т. д.

Понятно, что, напримѣръ, логическое выраженіе

$$a+b(c_0+d_0)+b_0d_0$$

означаетъ сложное событіе, которое наступаетъ: во 1-хъ, при наступленіи событія a; во 2-хъ, при совпаденіи событія b или съ событіемъ не-c, или-же съ событіемъ не-d; и наконецъ, въ 3-хъ, при совпаденіи событія не-b съ событіемъ не-d.

§ 3. Изъ Теоріи Вѣроятностей извѣстно, что вѣроятность ненаступленія событія равна единицѣ (достовѣрности) безъ вѣроятности его наступленія.

Если такъ, то

$$P(a_0)=1-P(a).$$

Точно такъже, напримѣръ:

$$P(b_0)=1-P(b)$$

и пр.

§ 4. Далѣе, изъ Теоріи Вѣроятностей извѣстно, что если два событія несовмѣстны, то вѣроятность, что случится то или другое изъ нихъ, равна суммѣ ихъ отдѣльныхъ вѣроятностей. Поэтому, если логическіе классы m и n дисъюнктны, т. е. не имѣютъ общихъ предметовъ, (причемъ $mn=0$), то

$$P(m+n)=P(m)+P(n).$$

Это правило примѣнимо къ какому угодно числу несовмѣстныхъ одно съ другимъ событій. Для возможности пользоваться этимъ правиломъ необходимо умѣть каждый логическій многочленъ

$$A+B+C+D+\dots$$

приводить къ дисъюнктному виду, т. е. къ виду

$$A+A_0B+A_0B_0C+A_0B_0C_0D+\dots,$$

гдѣ A_0 есть отрицаніе A, B_0—отрицаніе B и т. д. Оба написанные многочлена логически равнозначны, но отличаются тѣмъ, что къ первому изъ нихъ не примѣнимо предыдущее правило, тогда какъ ко второму примѣнимо.

Итакъ, каждое сложное событіе, имѣющее видъ суммы, мы всегда можемъ выразить такъ, что его вѣроятность разобьется на сумму вѣроятностей другихъ, болѣе простыхъ, событій. Напр., вѣроятность

$$P(A+B+C+D),$$

будучи приведена къ виду

$$P(A+A_0B+A_0B_0C+A_0B_0C_0D),$$

разбивается на сумму вѣроятностей:

$$P(A)+P(A_0B)+P(A_0B_0C)+P(A_0B_0C_0D).$$

§ 5. Затѣмъ, изъ Теоріи Вѣроятностей извѣстно, что если два и болѣе событія суть независимыя, то вѣроятность ихъ совпаденія равна произведенію ихъ отдѣльныхъ вѣроятностей. Это значитъ, что если $a, b, c, \ldots$ суть простыя событія, не связанныя между собою никакимъ логическимъ отношеніемъ, то

$$P(abc\ldots)=P(a)P(b)P(c)\ldots$$

§ 6. Если такъ, то вѣроятность приведеннаго къ дисъюнктному виду логическаго многочлена

$$A+A_0B+A_0B_0C+\ldots.,$$

не подчиненнаго никакимъ условіямъ, можетъ быть изображена такъ:

$$P(A)+P(A_0)P(B)+P(A_0)P(B_0)P(C)+\ldots.,$$

т. е. получается изъ выраженія многочлена простою замѣной классовъ $A, B, C, \ldots$ и ихъ отрицаній вѣроятностями тѣхъ и другихъ.

Отсюда видимъ, что абсолютная вѣроятность всякой отдѣльной логической функціи

$$f(a,b,c,д,...),$$

приведенной предварительно къ дисъюнктному виду, есть

$$f[P(a),P(b),P(c),...].$$

Въ первомъ изъ этихъ выраженій f означаетъ совокупность *логическихъ* дѣйствій надъ качеств. символами $a,b,c,...$; во второмъ то-же f означаетъ совокупность *алгебраическихъ* дѣйствій надъ количеств. символами $P(a),P(b),P(c),...$.

Примѣръ. Если вѣроятности простыхъ событій x и y суть $P(x)=p$, $P(y)=q$, то вѣроятность сложнаго событія xy_0+x_0y, уже имѣющаго дисъюнктный видъ, есть $p(1-q)+(1-p)q$. Вѣроятность-же сложнаго событія $x+y$, которое, по приведеніи къ дисъюнктному виду, есть $x+x_0y$, или $y+y_0x$ выразится такъ: $p+(1-p)q$, или $q+(1-q)p$.

Такъ дѣлается переходъ отъ выраженія отдѣльной логической функціи къ выраженію абсолютной ея вѣроятности.

§ 7. Понятно теперь, что для перехода отъ логическаго равенства $f=\varphi$ къ отношенію между вѣроятностями входящихъ туда классовъ, надо привести обѣ функціи f и φ къ дисъюнктному виду и затѣмъ замѣнить въ обѣихъ частяхъ равенства всѣ качественные символы $a,b,c,...$ символами количественными $P(a),P(b),...$

Для примѣра превратимъ логическое равенство

$$ab+cд=ac+bд$$

въ отношеніе между вѣроятностями, принимая $P(a)=p$, $P(b)=q,P(c)=r,P(д)=s$.

Надо привести къ дисъюнктному виду обѣ части исходнаго равенства. Имѣемъ:

$$ab + (ab)_0 c\partial = ac + (ac)_0 b\partial$$
$$ab + (a_0 + b_0)c\partial = ac + (a_0 + c_0)b\partial$$
$$ab + (a_0 + ab_0)c\partial = ac + (a_0 + ac_0)b\partial$$
$$ab + a_0 c\partial + ab_0 c\partial = ac + a_0 b\partial + ac_0 b\partial.$$

Въ послѣднемъ равенствѣ обѣ части состоятъ изъ членовъ, дисъюнктныхъ между собою, а потому, дѣлая отъ него переходъ къ отношенію между вѣроятностями, получимъ:

$$pq + (1-p)rs + p(1-q)rs = pr + (1-p)qs + p(1-r)qs.$$

§ 8. Хотя, такимъ образомъ, при операціяхъ надъ логическими равенствами мы можемъ въ любой моментъ сдѣлать переходъ къ отношеніямъ между вѣроятностями; однако, при рѣшеніи задачи объ опредѣленіи вѣроятности одного событія посредствомъ вѣроятностей другихъ событій, всего натуральнѣе поступать такъ: найти изъ всей совокупности данныхъ логич. условій опредѣленіе перваго событія помощію остальныхъ и уже затѣмъ сдѣлать переходъ къ вѣроятностямъ. Этого пріема мы и будемъ держаться.

§ 9. Доселѣ мы вели рѣчь объ абсолютныхъ вѣроятностяхъ. Обращаемся къ вѣроятностямъ относительнымъ.

Въ Теоріи Вѣроятностей доказывается слѣдующая истина: вѣроятность, что если событіе A случится, то и событіе B тоже случится, равна вѣроятности совпаденія событій A и B, раздѣленной на вѣроятность событія A, т. е. равна дроби $\frac{P(AB)}{P(A)}$.

Поэтому, если $A = f(a,b,c,\partial,\ldots)$, $B = \varphi(a,b,c,\partial,\ldots)$, то искомая относительная вѣроятность получится, если въ выраженіи произведенія f и φ, приведеннаго къ дисъюнктному виду, замѣнимъ $a,b,c,\ldots$ ихъ абсолютными вѣроятностями и полученный результатъ раздѣлимъ на выраженіе функціи f, приведенное къ дисъюнктному виду, причемъ въ немъ надо также замѣнить всѣ качественные символы количественными. И такъ, искомая относит. вѣроятность будетъ:

$$\frac{[f(a,b,c,\ldots)\varphi(a,b,c,d\ldots)]}{[f(a,b,c,d\ldots)]},$$

гдѣ заключеніе въ прямыя скобки означаетъ упомянутую замѣну.

Для примѣра, полагая $P(x)=p, P(y)=q, P(z)=r$, найдемъ вѣроятность, что если случится событіе

$$xy_0+x_0y,$$

т. е. одно изъ событій x и y, но не оба вмѣстѣ, то случится также и событіе

$$yz_0+y_0z,$$

т. е. одно изъ событій y и z, но не оба вмѣстѣ.

Въ данномъ случаѣ

$$f(x,y,z)=xy_0+x_0y,\ \varphi(x,y,z)=yz_0+y_0z$$
$$f(x,y,z)\varphi(x,y,z)=xy_0z+x_0yz_0.$$

Слѣд. искомая относит. вѣроятность есть:

$$\frac{[f\varphi]}{[f]}=\frac{p(1-q)r+(1-p)q(1-r)}{p(1-q)+q(1-p)}.$$

§ 10. Предположимъ теперь, что даны относительныя вѣроятности простыхъ событій $a,b,c,\ldots$, высчитанныя такъ, чтобы удовлетворялся рядъ условій

$$f'(a,b,c,\ldots)=\varphi'(a,b,c,\ldots),\ f''=\varphi'',\ f'''=\varphi''',\ldots,$$

и требуется найти абсолютныя вѣроятности тѣхъ-же простыхъ событій.

Замѣтимъ, прежде всего, что каждое логическое равенство

$$f(a,b,c,...)=\varphi(a,b,c,...)$$

можетъ быть тождественно замѣнено равенствомъ

$$1=f\varphi+f_0\varphi_0,$$

гдѣ 1 означаетъ логическій міръ рѣчи (въ данномъ случаѣ, міръ всѣхъ событій, о которыхъ идетъ рѣчь), f_0 и φ_0 суть отрицанія f и φ.

Кромѣ того, извѣстно, что вся совокупность данныхъ условій вполнѣ равнозначна съ однимъ условіемъ:

$$1=(f'\varphi'+f'_0\varphi'_0)(f''\varphi''+f''_0\varphi''_0)(f'''\varphi'''+f'''_0\varphi'''_0)...,$$

которое можно сокращенно представить подъ формой

$$1=M(a,b,c,д....).$$

Въ этомъ равенствѣ, тождественно замѣняющемъ всѣ данныя условія, функція M называется логическимъ міромъ рѣчи задачи или полной логич. единицей задачи.

И такъ, подчиненіе классовъ $a,b,c,...$ всей совокупности исходныхъ условій вполнѣ равнозначно съ подчиненіемъ ихъ одному условію $1=M(a,b,c,д...)$, составленному по правилу указанному выше.

Пусть теперь $p,q,r,..$ означаютъ вѣроятности событій $a,b,c,...$, подчиненныхъ условію $1=M$, и пусть $p',q',r',...$ —абсолютныя вѣроятности тѣхъ-же событій. Такъ какъ первыя изъ этихъ вѣроятностей означаютъ вѣроятности, что при наступленіи событія M (міра рѣчи) случатся событія $a,b,c,...$, то, по доказанному ранѣе, для опредѣленія абсолютныхъ вѣроятностей $p',q',r',..$, будемъ имѣть:

$$p=\frac{[aM]}{[M]},\ q=\frac{[bM]}{[M]},\ r=\frac{[cM]}{[M]},\ldots,$$

гдѣ въ правыхъ частяхъ классы $a,b,c,\ldots$ должны быть замѣнены ихъ абсолютными вѣроятностями $p',q',r',\ldots$, которыя и найдутся чрезъ рѣшеніе системы полученныхъ алгебраич. уравненій.

Возьмемъ примѣръ. Пусть при выниманіи изъ урны шаровъ обращали вниманіе только на случаи, когда вынутый шаръ былъ или бѣлый, или мраморный (или то и другое вмѣстѣ), и пусть, при этомъ условіи, найдены: p—вѣроятность бѣлаго шара, q—мраморнаго. Найти абсолютныя ихъ вѣроятности p' и q'.

Построимъ сначала условіе, съ подчиненіемъ которому были найдены вѣроятности p и q. Пусть x есть вынутіе бѣлаго шара, y—мраморнаго. Если при высчитываніи вѣроятностей исключались случаи, когда вынутый шаръ былъ не бѣлый и не мраморный, то это значитъ, что было соблюдаемо условіе:

$$x_0 y_0=0,$$

или, что тоже:

$$1=xy+x_0y+xy_0.$$

И такъ, въ данномъ случаѣ

$$\begin{aligned} M(x,y)&=xy+x_0y+xy_0\\ xM(x,y)&=xy+xy_0=x\\ yM(x,y)&=xy+x_0y=y. \end{aligned}$$

А потому имѣемъ:

$$p=\frac{[Mx]_{x=p',y=q'}}{[M]_{x=p',y=q'}},\ q=\frac{[My]_{x=p',y=q'}}{[M]_{x=p',y=q'}},$$

или:

$$p=\frac{p'}{p'q'+p'(1-q')+q'(1-p')},\ q=\frac{q'}{p'q'+p'(1-q')+q'(1-p')}.$$

Чрезъ рѣшеніе этихъ двухъ алгебр. уравненій получимъ

$$p'=\frac{p+q-1}{q},\ \ q'=\frac{p+q-1}{p}.$$

§ 11. Согласно съ тѣмъ, что высказано ранѣе, для опредѣленія вѣроятности одного событія черезъ вѣроятности другихъ событій, намъ надо прежде всего логически выразить первое черезъ остальныя. Это насъ заставляетъ сказать нѣсколько словъ о пріемахъ опредѣленія одного логическаго класса (простаго или сложнаго) черезъ всѣ или нѣкоторые изъ прочихъ.

Пусть требуется опредѣлить простой классъ a черезъ всѣ прочіе классы $b,c,д,\ldots$, связанные съ a и между собою рядомъ условій (посылокъ)

$$f'=\varphi',\ f''=\varphi'';\ f'''=\varphi''',\ldots$$

Всѣ эти условія тождественно могутъ быть замѣнены однимъ:

$$1=M(a,b,c,д,\ldots).$$

Съ другой стороны, это послѣднее равенство можетъ быть тождественно замѣщено слѣдующими тремя:

$$a=aM(1,b,c,д\ldots)=aM(1)$$
$$a=a+M(1,b,c,\ldots)M_0(0,b,c,\ldots)=a+M(1)M_0(0)$$
$$1=M(1,b,c,\ldots)+M(0,b,c,д\ldots)=M(1)+M(0).$$

Здѣсь $M(1)$ есть результатъ замѣщенія въ функціи

$M(a,b,c,...)$ класса a единицей, а его отрицанія a_0 нулемъ, $M(0)$ есть результатъ замѣщенія въ $M(a,b,c...)$ класса a нулемъ, а его отрицанія a_0 единицей; $M_0(0)$ есть отрицаніе функціи $M(0)$ или, что тоже, результатъ замѣщенія въ отрицаніи функціи M, т. е. въ функціи $M_0(a,b,c,...)$, класса a нулемъ, а его отрицанія a_0 единицей.

Изъ послѣднихъ трехъ равенствъ первое показываетъ, что a содержится въ $M(1)$, второе—что a содержитъ въ себѣ $M_0(0)M(1)$. Вотъ почему эти два равенства можно замѣнить неравенствами

$$a < M(1),\ a > M_0(0)M(1),$$

которыя надо понимать въ смыслѣ: a не больше $M(1)$ и не меньше $M_0(0)M(1)$.

Наконецъ, третье равенство $1 = M(1) + M(0)$, зависящее отъ классовъ $b,c,d,...$, но не содержащее класса a, представляетъ условіе, которому, въ силу первоначальныхъ условій, подчинены тѣ двѣ функціи $M(1)$ и $M_0(0)M(1)$, помощію которыхъ опредѣляется a.

Въ случаѣ, когда эти двѣ функціи логически равнозначны, т. е. когда

$$M_0(0)M(1) = M(1),$$

два неравенства, опредѣляющія a, суть:

$$a > M(1),\ a < M(1),$$

т. е. доставляютъ одно равенство:

$$a = M(1).$$

Если желаемъ опредѣлить a изъ того-же уравненія $1=$

$M(a,b,c,...)$ не черезъ всѣ, но черезъ нѣкоторые изъ классовъ $b,c,д...$, то всѣ лишніе классы надо исключить изъ равенства $1=M(a,b,c,д...)$. Для этого исключенія достаточно замѣнить въ равенствѣ $1=M(a,b,c,...)$ всѣ исключаемые классы, а также ихъ отрицанія, единицами. Пусть результатъ исключенія будетъ: $1=M'$, гдѣ M' зависитъ отъ a и нѣкоторыхъ изъ прочихъ классовъ. Затѣмъ останется опредѣлить a изъ равенства $1=M'$ совершенно такъ, какъ мы выше опредѣляли его изъ равенства $1=M$.

Такъ опредѣляется простой классъ черезъ всѣ или нѣкоторые прочіе простые классы на основаніи какого бы то ни было числа данныхъ логическихъ условій.

§ 12. Обращаемся къ опредѣленію сложныхъ классовъ, т. е. функцій.

Легко показать, что логическая функція можетъ быть выражена черезъ простые классы (всѣ или нѣкоторые) даже тогда, когда эти послѣдніе не подчинены никакимъ условнымъ равенствамъ.

Въ самомъ дѣлѣ, пусть даны n простые классы $a,b,c,...$, не связанные между собою никакими условіями, и сложный классъ A, гдѣ A означаетъ опредѣленную функцію тѣхъ-же классовъ. Въ такомъ случаѣ, положивъ $A=w$, или, что тоже, $1=Aw+A_0w_0$, можемъ сказать, что мы имѣемъ $n+1$ простыхъ классовъ: $w,a,b,c,....$, которые подчинены условію

$$1=Aw+A_0w_0=M(w,a,b,c,...).$$

Изъ этого условія и можетъ быть опредѣленъ простой классъ w (т. е. функція A) черезъ всѣ или нѣкоторые изъ прочихъ классовъ по правиламъ, указаннымъ выше.

Такимъ образомъ, разсматриваніе хотя-бы только одной логической функціи совмѣстно съ независимыми простыми классами обращаетъ задачу изъ безусловной въ условную.

Если, рядомъ съ n независимыми простыми классами $a, b, c, \ldots$, мы начнемъ разсматривать m функцій $U, V, W, \ldots$, то, введя рядъ обозначеній

$$U = u,\ \ V = v,\ \ W = w, \ldots,$$

мы получаемъ задачу объ $n+m$ простыхъ классахъ: $a, b, c \ldots$, $u, v, w \ldots$, подчиненныхъ условію:

$$\begin{aligned} 1 = (uU + u_0 U_0)(vV + v_0 V_0)(wW + w_0 W_0)\ldots \\ = M(a, b, c, \ldots, u, v, w, \ldots), \end{aligned}$$

изъ котораго по предъидущему и можетъ быть логически опредѣленъ любой изъ классовъ $u, v, w, \ldots$ помощію всѣхъ или нѣкоторыхъ изъ прочихъ классовъ, т. е. найдется любая изъ функцій $U, V, W, \ldots$ помощію всѣхъ или нѣкоторыхъ изъ данныхъ простыхъ классовъ и всѣхъ или нѣкоторыхъ изъ прочихъ функцій.

Наконецъ, если простые n классы $a, b, c, д \ldots$ суть зависимые, связанные между собою p условіями

$$A' = B',\ A'' = B'',\ A''' = C''', \ldots,$$

гдѣ $A', B', A'', B'', \ldots$ суть функціи $a, b, c, д, \ldots$, то при опредѣленіи одной изъ ряда m функцій

$$U, V, W, \ldots$$

мы будемъ имѣть задачу объ $n+m$ простыхъ классахъ: a, b, $c, д, \ldots, u, v, w, \ldots$, связанныхъ между собою $p+m$ условіями

$$A' = B',\ A'' = B'', \ldots,\ u = U,\ v = V,\ w = W, \ldots$$

или, что тоже, однимъ условіемъ:

$$1=(A'B'+A'_0B'_0)(A''B''+A''_0B''_0)\ldots(uU+u_0U_0)(vV+v_0V_0)\ldots,$$

которое можно изобразить такъ:

$$1=M(a,b,c,д,\ldots,u,v,w,\ldots).$$

Отсюда и можетъ быть найдена по предыдущему любая изъ функцій $U, V, W,\ldots$ помощію всѣхъ или нѣкоторыхъ изъ прочихъ функцій, а также всѣхъ или нѣкоторыхъ изъ простыхъ классовъ $a,b,c,д,\ldots$, причемъ всѣ исходныя условныя равенства будутъ приняты во вниманіе.

§ 13. Вотъ мы имѣемъ всѣ данныя для рѣшенія поставленной въ началѣ статьи общей задачи объ опредѣленіи вѣроятности одной функціи (одного сложнаго событія) посредствомъ вѣроятностей всѣхъ или нѣкоторыхъ прочихъ функцій и простыхъ классовъ, предполагая, что послѣдніе связаны между собою какимъ-бы то ни было числомъ условныхъ равенствъ.

Пусть, поступая по предыдущему, мы пришли къ равенству

$$1=M(a,b,c,\ldots u,v,w,\ldots),$$

изъ котораго уже исключены всѣ классы и функціи, вѣроятности которыхъ не должны быть принимаемы во вниманіе при опредѣленіи вѣроятности функціи U помощію вѣроятностей прочихъ классовъ $a,b,c,\ldots,v,w,\ldots$

Въ такомъ случаѣ мы получимъ:

$$u<M(1),\quad u>M_0(0)M(1),$$

гдѣ $M(1)$ и $M(0)$ суть результаты замѣщенія въ функціи M класса u единицей и нулемъ соотвѣтственно, (а его отрицанія нулемъ и единицей), причемъ между прочими классами $a,b,c,\ldots v,w,\ldots$ устанавливается отношеніе:

$$1 = M(1) + M(0) = K.$$

Остается перейти къ опредѣленію вѣроятности u. Пусть вѣроятности классовъ $a, b, c, ..v, w, ...$, найденныя съ соблюденіемъ всѣхъ первоначальныхъ условій задачи, а слѣдовательно также подчиненныя и условію $1 = K$, суть $p, q, r, ...$ $\alpha, \beta, ..$ Въ такомъ случаѣ ихъ абсолютныя вѣроятности, которыя мы назовемъ черезъ $p', q', r', ...\alpha', \beta', ...$, надо искать изъ условій:

$$p = \frac{[aK]}{[K]}, q = \frac{[bK]}{[K]}, ..., \alpha = \frac{[vK]}{[K]}, \beta = \frac{[wK]}{[K]}, ...,$$

гдѣ въ правыхъ частяхъ, по приведеніи числителей и знаменателей къ дисъюнктному виду, всѣ качественные символы $a, b, c, ...v, w, ...$ должны быть замѣнены количественными символами $p', q', r', ...\alpha', \beta', ...$

Найденныя отсюда величины $p', q', r', ...\alpha', \beta', ...$, будучи подставлены, вмѣсто $a, b, c, .. , v, w, ...$ въ правыя части неравенствъ

$$u < M(1), \quad u > M_0(0) M(1),$$

доставятъ намъ абсолютныя вѣроятности функцій $M(1)$ и $M_0(0)M(1)$, т. е. предѣлы для абсолютной вѣроятности функціи u.

Однако, намъ нужно знать не абсолютную, но относительную вѣроятность функціи u, а именно такую, въ которой были бы приняты во вниманіе всѣ условныя равенства задачи, а слѣд. также и условіе $1 = K$. Въ силу доказаннаго ранѣе, такого рода относительныя вѣроятности функцій $M(1)$ и $M_0(0)M(1)$ суть соотвѣтственно:

$$\frac{[M(1)K]}{[K]}, \quad \frac{[M_0(0)M(1)K]}{[K]},$$

гдѣ всѣ качественные символы $a,b,c...,v,w,...$ должны быть замѣнены соотвѣтственными абсолютными вѣроятностями p', $q',r',...,\alpha',\beta',...$ Но

$$K = M(1) + M(0),$$

а потому

$$M(1)K = M(1)[M(1) + M(0)] = M(1)$$
$$M_0(0)M(1)K = M_0(0)M(1)[M(1) + M(0)] = M_0(0)M(1).$$

Слѣдовательно, относительныя вѣроятности функцій $M(1)$ и $M_0(0)M(1)$ суть

$$\frac{[M(1)]}{[K]} \text{ и } \frac{[M_0(0)M(1)]}{[K]}.$$

Если такъ, то, называя искомую относительную вѣроятность функціи u черезъ $P(u)$, получимъ

$$(1) \qquad P(u) < \frac{[M(1)]}{[K]}, \quad P(u) > \frac{[M_0(0)M(1)]}{[K]},$$

гдѣ всѣ качественные символы $a,b,c,...v,w,...$ должны быть замѣнены символами $p',q',r',...\alpha',\beta',...$ Послѣ такой замѣны вмѣсто этихъ послѣднихъ символовъ должны быть подставлены ихъ значенія, выраженныя помощію $p,q,r,...\alpha,\beta,...$ на основаніи равенствъ:

$$(2) \qquad p = \frac{[aK]}{[K]}, q = \frac{[bK]}{[K]}, ..., \quad \alpha = \frac{[vK]}{[K]}, \beta = \frac{[wK]}{[K]}, ...,$$

въ которыхъ предварительно должно быть сдѣлано то-же замѣщеніе символовъ $a,b,c,...v,w,...$ символами $p',q',r',...\alpha',\beta',...$ Но если въ формулахъ (1) и (2) качественные символы a, $b,c,...,v,w,...$ замѣняются количественными символами $p',q',r',...$,

$\alpha', \beta', \ldots$, которые вслѣдъ затѣмъ исключаются изъ (1) помощію (2), то понятно, что нѣтъ надобности дѣлать означенное замѣщеніе на самомъ дѣлѣ, а совершенно достаточно начать считать въ (1) и (2) качественные символы $a, b, c, \ldots$ $v, w, \ldots$ какъ-бы количественными и исключить ихъ, по правиламъ Алгебры, изъ (1) помощію (2). Такимъ образомъ, окончательная форма рѣшенія задачи объ опредѣленіи $P(u)$ помощію относит. вѣроятностей $p, q, r, \ldots, \alpha, \beta, \ldots$ есть такова: помощію равенствъ:

$$K = M(1) + M(0) = \frac{aK}{p} = \frac{bK}{q} = \ldots = \frac{vK}{\alpha} = \frac{wK}{\beta} = \ldots,$$

гдѣ, по приведеніи всѣхъ многочленовъ къ дисъюнктному виду, символы $a, b, c, \ldots v, w, \ldots$ принимаются алгебраическими, исключить всѣ эти символы изъ пары неравенствъ:

$$P(u) < \frac{M(1)}{K}, \quad P(u) > \frac{M_0(0) M(1)}{K},$$

въ которыхъ тоже всѣ многочлены должны быть приведены къ дисъюнктному виду, символы-же $a, b, c, \ldots v, w, \ldots$ трактуются количественными.

Таковъ общій способъ рѣшенія задачи, формулированной въ началѣ статьи. Какъ видимъ, вообще для искомой вѣроятности $P(u)$ получаются только предѣлы, между которыми она содержится; и только тогда, когда

$$M_0(0) M(1) = M(1),$$

получается точное опредѣленіе $P(u)$, именно:

$$P(u) = \frac{M(1)}{K}.$$

§ 14. Обращаемся къ примѣрамъ.

Примѣръ 1-й. Пусть вѣроятность, что умретъ въ такомъ-то году или A, или B, (или оба), есть p; вѣроятность, что не умретъ въ томъ же году или A, или B (или оба), есть q. Найти вѣроятность, что умретъ въ томъ-же году только одинъ изъ нихъ (т. е. или A, при чемъ B останется живъ, или обратно).

Пусть x событіе смерти A, y—смерти B.

Даны: $P(x+y)=p$, $P(x_0+y_0)=q$. Ищется $P(xy_0+x_0y)$.

Здѣсь мы имѣемъ 3 функціи. Положимъ:

$$x+y=s,\ x_0+y_0=t,\ xy_0+yx_0=w.$$

Задачу можно считать содержащей пять простыхъ классовъ, связанныхъ этими тремя условіями, или, что тоже, однимъ слѣдующимъ:

$$\begin{aligned}1=&[s(x+y)+s_0x_0y_0][t(x_0+y_0)+t_0xy]\times\\ &\times[w(xy_0+x_0y)+w_0(x_0y_0+xy)]=\\ &=stwxy_0+stwx_0y+st_0w_0xy+s_0tw_0x_0y_0.\end{aligned}$$

Намъ надо найти изъ этого равенства выраженіе для w черезъ s и t; лишніе классы x и y надо исключить (что достигается подстановкою: $x=1$, $y=1$, $x_0=1$, $y_0=1$). Результатъ этого исключенія есть:

$$1=M(s,t,w)=stw+st_0w_0+s_0tw_0=M(w),$$

откуда

$$\begin{gathered}M(1)=st,\ M(0)=st_0+s_0t,\ M_0(0)=st+s_0t_0,\\ M_0(0)M(1)=st,\ K=M(1)+M(0)=s+s_0t,\\ Ks=s,\ Kt=ts+ts_0=t.\end{gathered}$$

Такъ какъ въ данномъ случаѣ $M_0(0)M(1)$ равно $M(1)$, то два неравенства, опредѣляющія функцію w, сводятся на одно равенство

$$w = M(1) = st.$$

И дѣйствительно, произведеніе $s = x + y$ на $t = x_0 + y_0$ есть $w = xy_0 + x_0 y$.

И такъ, искомая вѣроятность $P(w)$ опредѣлится равенствомъ

$$P(w) = \frac{M(1)}{K} = \frac{st}{s + s_0 t},$$

послѣ исключенія изъ него, считаемыхъ количественными, символовъ s и t помощію равенствъ:

$$K = s + s_0 t = \frac{s}{p} = \frac{t}{q}.$$

Изъ этихъ равенствъ имѣемъ:

$$p = \frac{s}{K},\ q = \frac{t}{K},\ p + q = \frac{s+t}{K},\ p + q - 1 = \frac{s+t-K}{K} =$$
$$= \frac{s + t - (s + (1-s)t)}{K} = \frac{t - t + ts}{K} = \frac{ts}{K}.$$

Слѣдовательно, окончательно:

$$P(w) = p + q - 1.$$

Для повѣрки замѣтимъ слѣдующее. Если $P(x+y) = p$ то $P[(x+y)_0] = P(x_0 y_0) = 1 - p$. Точно такъ-же, если $P(x_0 + y_0) = q$, то $P(xy) = 1 - q$.

Слѣд.

$$P(xy + x_0 y_0) = P(xy) + P(x_0 y_0) = 2 - p - q,$$

а потому

$$P(xy_0+x_0y)=P[(xy+x_0y_0)_0]=1-[2-p-q]=p+q-1,$$

результатъ, вполнѣ согласный съ найденнымъ выше.

Примѣръ 2-ой. Пусть вѣроятность, что свидѣтель A показываетъ истину, есть p; вѣроятность, что свидѣтель B показываетъ истину, есть q; вѣроятность несовпаденія ихъ показаній есть r. Найти вѣроятность, что если ихъ показанія совпадаютъ, то получается истина.

Пусть классы случаевъ, когда свидѣтели A и B соотвѣтственно показываютъ истину, суть x и y. Даны:

$$P(x)=p,\ P(y)=q,\ P(xy_0+x_0y)=r.$$

Ищется отношеніе

$$\frac{P(xy)}{P(xy+x_0y_0)}=\frac{P(xy)}{1-r}.$$

Очевидно, достаточно найти только $P(xy)$ посредствомъ p,q и r.

Пусть

$$xy_0+x_0y=s,\ xy=w.$$

Совокупность этихъ двухъ условій равнозначна съ однимъ равенствомъ:

$$1=ws_0xy+w_0(sx_0y+s_0x_0y_0+sxy_0).$$

Вотъ какому условію подчинена данная задача о четырехъ простыхъ классахъ x,y,s,w. Требуется опредѣлить w черезъ всѣ прочіе классы. Имѣемъ:

$$1=ws_0xy+w_0(sx_0y+s_0x_0y_0+sxy_0)=M(w),$$
$$M(1)=s_0xy,\ M(0)=s(x_0y+xy_0)+s_0x_0y_0,$$

$$M_0(0)=s(xy+x_0y_0)+s_0(x+y),\ M_0(0)M(1)=s_0xy=M(1),$$
$$K=M(1)+M(0)=s_0xy+s_0x_0y_0+sx_0y+sxy_0.$$

Такъ какъ $M_0(0)M(1)=M(1)=s_0xy$, то, вмѣсто двухъ неравенствъ, w опредѣляется однимъ равенствомъ

$$w=s_0xy.$$

Кромѣ того,

$$Kx=s_0xy+sxy_0,\ Ky=s_0xy+sx_0y,\ Ks=sx_0y+sxy_0.$$

Считая x, y и s количественными символами, намъ надо исключить ихъ изъ формулы:

$$P(w)=\frac{s_0xy}{K}$$

помощію отношеній:

$$\frac{xys_0+xy_0s}{p}=\frac{xys_0+x_0ys}{q}=\frac{x_0ys+xy_0s}{r}=K=s_0xy+s_0x_0y_0+sx_0y+sxy_0.$$

Имѣемъ:

$$r=\frac{sx_0y}{K}+\frac{sxy_0}{K},$$

$$q=\frac{s_0xy}{K}+\frac{sx_0y}{K},$$

$$p=\frac{s_0xy}{K}+\frac{sxy_0}{K}=\frac{s_0xy}{K}+\left(r-\frac{sx_0y}{K}\right)=\frac{s_0xy}{K}+r+\frac{s_0xy}{K}-q.$$

Слѣд.

$$\frac{s_0xy}{K}=\frac{p+q-r}{2}.$$

А потому окончательно:

$$P(w)=\frac{p+q-r}{2},$$

$$\frac{P(xy)}{P(xy+x_0y_0)}=\frac{p+q-r}{2(1-r)}.$$

Примѣръ 3-й. Пусть изъ наблюденій относительно эпидемій въ какой-нибудь мѣстности найдено, что p есть вѣроятность посѣщенія отдѣльнаго дома горячкой, q—холерой, r есть вѣроятность непосѣщенія дома обѣими болѣзнями при удовлетворительности санитарныхъ его условій.

Найти вѣроятность неудовлетворительности санитарныхъ условій отдѣльнаго дома въ той-же мѣстности.

Пусть x—посѣщеніе дома горячкой, y—холерой, z—неудовлетворительность санитарныхъ условій дома. Даны:

$$P(x)=p,\ P(y)=q,\ P(x_0y_0z_0)=r.$$

Найти $P(z)$.

Пусть

$$x_0y_0z_0=w.$$

Условіе, которому подчинена данная задача о четырехъ простыхъ классахъ x,y,z,w, есть:

$$1=wx_0y_0z_0+w_0(x+y+z)=F(z).$$

Отсюда надо найти z посредствомъ x,y,w.

Имѣемъ:

$$F(1)=w_0,\ F(0)=wx_0y_0+w_0(x+y),$$
$$F_0(0)=w(x+y)+w_0x_0y_0,\ F_0(0)F(1)=w_0x_0y_0.$$

Слѣдовательно

$$z<w_0,\ z>w_0x_0y_0.$$

Кромѣ того,

$$K=F(1)+F(0)=w_0+wx_0y_0+w_0(x+y)=w_0+wx_0y_0,$$

$$Kx = xw_0,\ Ky = yw_0,\ Kw = wx_0y_0.$$

Надо исключить, считаемые количественными, символы w, x, y изъ неравенствъ

$$P(z) < \frac{w_0}{K},\ P(z) > \frac{w_0x_0y_0}{K}$$

помощію отношеній:

$$\frac{xw_0}{p} = \frac{yw_0}{q} = \frac{wx_0y_0}{r} = K = w_0 + wx_0y_0.$$

Имѣемъ:

$$w_0 = \frac{wx_0y_0}{r} - wx_0y_0 = \frac{wx_0y_0(1-r)}{r} = K(1-r);\ \frac{w_0}{K} = 1-r;$$

$$p+r = \frac{xw_0 + wx_0y_0}{K};\ 1-p-r = \frac{K-xw_0-wx_0y_0}{K} = \frac{w_0-xw_0}{K} = \frac{x_0w_0}{K};$$

$$q+r = \frac{yw_0 + wx_0y_0}{K};\ 1-q-r = \frac{K-yw_0-wx_0y_0}{K} = \frac{w_0-yw_0}{K} = \frac{y_0w_0}{K};$$

$$(1-p-r)(1-q-r) = \frac{w_0^2x_0y_0}{K^2};$$

$$\frac{(1-p-r)(1-q-r)}{1-r} = \frac{w_0^2x_0y_0}{K^2} \cdot \frac{K}{w_0} = \frac{w_0x_0y_0}{K}.$$

А потому окончательно:

$$P(z) < 1-r,\ P(z) > \frac{(1-p-r)(1-q-r)}{1-r}.$$

Примѣръ 4-й. Пусть относительно шаровъ, находящихся въ данной урнѣ, извѣстно, что всякій бѣлый шаръ есть или крупный, или немраморный. Пусть при выниманіи шаровъ изъ этой урны обращается вниманіе только на такіе случаи, когда вынутый шаръ есть или бѣлый, или крупный, или мраморный. Пусть при этихъ условіяхъ найдено для вѣроятности случая, когда вынутый шаръ есть и бѣлый, и крупный, число p. Найти вѣроятность, что будетъ вынутъ шаръ или бѣлый, но некрупный, или, если не бѣлый, то или крупный, или-же мраморный.

Пусть x—вынутіе бѣлаго шара, y—крупнаго, z—мраморнаго.

Первоначальныя два условія задачи суть:

$$x = x(y + z_0)$$
$$1 = x + y + z.$$

Дана вѣроятность $P(xy) = p$. Ищется вѣроятность $P(xy_0 + x_0(y + z))$.

Положимъ

$$xy = u,\ \ xy_0 + x_0(y + z) = v.$$

Можно сказать, что данная задача содержитъ 5 простыхъ классовъ: x, y, z, u, v, подчиненыхъ всѣмъ, написаннымъ выше, четыремъ условіямъ. Всѣ эти условія совмѣщаются въ одно слѣдующее:

$$1 = [x_0 + y + z_0][x + y + z][uxy + u_0x_0 + u_0y_0] \times$$
$$\times [vxy_0 + vx_0y + vx_0z + v_0xy + v_0x_0y_0z_0] =$$
$$= uv_0xy + u_0vx_0y + u_0vxz_0 + u_0vxy_0z_0.$$

По смыслу задачи, отсюда требуется опредѣлить v черезъ u. Лишніе классы: x, y, z должны быть исключены, что достигается подстановкою:

$$x = y = z = x_0 = y_0 = z_0 = 1.$$

По исключеніи получимъ:

$$1 = uv_0 + u_0v = F(v).$$

Отсюда имѣемъ:

$$F(1)=u_0,\ F(0)=u,\ F_0(0)=u_0,\ F_0(0)F(1)=u_0.$$

Слѣд. въ данномъ случаѣ v опредѣляется равенствомъ

$$v=u_0.$$

Далѣе, имѣемъ:

$$K=F(1)+F(0)=u_0+u=1.$$

Слѣдовательно, условіе $1=K$, которому подчинена функція u, сводится на тождество $1=1$, что равнозначно съ отсутствіемъ всякаго условія. А потому получимъ окончательно:

$$P(v)=P(u_0)=1-p$$

ПРИЛОЖЕНІЕ.

О НУМЕРИЗАЦІИ ЛОГИЧЕСКИХЪ РАВЕНСТВЪ ВООБЩЕ.

Выше (§ 1) было показано, что каждому логическому равенству

(1) $$f(a,b,c,..)=\varphi(a,b,c,...)$$

соотвѣтствуетъ количественное равенство:

(2) $$N[f(a,b,c,...)]=N[\varphi(a,b,c,...)],$$

выражающее равенство чиселъ предметовъ, содержащихся въ классахъ f и φ.

Чрезъ дѣленіе обѣихъ частей этого послѣдняго равенства на число $N(1)$, означающее число предметовъ міра рѣчи, получается еще одно числовое равенство

(3) $$P[f(a,b,c,...)]=P[\varphi(a,b,c,...)],$$

выражающее равенство вѣроятностей логическихъ классовъ f и φ.

Назовемъ для краткости переходъ отъ равенства (1) къ равенству (3) *пробабилизаціей* логическаго равенства (1); переходъ-же отъ равенства (1) къ равенству (2)—*нумеризаціей* логическаго равенства (1).

Выше мы занимались *непосредственной* пробабилизаціей логическаго равенства, дѣлая переходъ прямо отъ равенства (1) къ равенству (3), безъ посредства промежуточнаго равенства (2). При этомъ для установленія свойствъ символа P намъ было необходимо пользоваться нѣкоторыми истинами Теоріи Вѣроятностей.

Но если мы построимъ правила для перехода отъ равенства (1) къ равенству (2), причемъ при установленіи свойствъ символа N уже нельзя будетъ пользоваться истинами Теоріи Вѣроятностей, то, въ виду простой связи между равенствами (2) и (3), въ правилахъ этихъ мы получимъ вмѣстѣ съ тѣмъ новый способъ опредѣленія нѣкоторыхъ свойствъ символа P.

Слѣдуетъ также замѣтить, что равенство (2) можетъ имѣть значеніе не только въ качествѣ промежуточнаго между (1) и (3), но и само по себѣ, такъ какъ оно можетъ найти себѣ примѣненіе въ другихъ областяхъ знаній, напр. въ Статистикѣ.

Обращаюсь къ построенію правилъ нумеризаціи логическихъ равенствъ.

Для нумеризаціи логич. равенства достаточно нумеризировать каждую его часть порознь и затѣмъ приравнять между собою результаты. Такимъ образомъ, нумеризація логическихъ равенствъ сводится къ нумеризаціи отдѣльныхъ логическихъ функцій.

Опредѣленіе числа предметовъ, содержащихся въ каждомъ логич. классѣ a, т. е. числа $N(a)$, можетъ быть достигнуто помощію непосредственнаго ихъ счета на самомъ дѣлѣ. Однако, зная зависимость между нѣкоторыми изъ символовъ $N(a)$, $N(b)$, $N(a+b)$, $N(ab)$ и пр., мы можемъ опредѣлять величину однихъ изъ этихъ символовъ по даннымъ величинамъ другихъ.

Установленіе разныхъ видовъ зависимостей между различными символами N и составляетъ предметъ теоріи нумеризаціи.

Найдемъ сначала отношеніе между двумя символами $N[f_0(a,b,c...)]$ и $N[f(a,b,c,...)]$, гдѣ f_0 есть логическое отрицаніе f.

Изъ логическаго тождества

$$f(a,b,c,...)+f_0(a,b,c...)=1$$

имѣемъ:

$$N[f(a,b,c...)+f_0(a,b,c,...)]=N(1).$$

Но такъ какъ произведеніе ff_0 равно нулю, то всѣ предметы функціи f отличны отъ предметовъ функціи f_0, а потому

$$N[f+f_0]=N(f)+N(f_0)$$

и слѣдовательно

$$N(f)+N(f_0)=N(1),$$

откуда

$$N[f_0(a,b,c...)]=N(1)-N[f(a,b,c,...)].$$

Это и есть искомое отношеніе. Дѣля въ немъ обѣ части на $N(1)$, получимъ отношеніе

$$P[f_0(a,b,c,....)]=1-P[f(a,b,c,.,)],$$

т. е. одну изъ основныхъ истинъ Теоріи Вѣроятностей.

Найдемъ выраженіе для символа $N(a+b)$.

Если a и b дисъюнктны, т. е. $ab=0$, то понятно, что

$$N(a+b)=N(a)+N(b).$$

Но пусть a и b конъюнктны, т. е. ab отлично отъ нуля. Изъ логическаго тождества

$$a=ab+ab_0,$$

гдѣ въ правой части оба члена дисъюнктны, получаемъ

$$N(a)=N(ab)+N(ab_0).$$

Точно также изъ тождества

$$b=ab+a_0b,$$

гдѣ опять оба члена правой части дисъюнктны, находимъ

$$N(b)=N(ab)+N(a_0b).$$

Складывая выраженія для $N(a)$ и $N(b)$, будемъ имѣть:

$$N(a)+N(b)=2N(ab)+N(ab_0)+N(a_0b).$$

Съ другой стороны, сумма предъидущихъ выраженій для a и b доставляетъ намъ (на основаніи общаго закона логики $m+m=m$) логическое равенство:

$$a+b=ab+ab_0+a_0b,$$

въ которомъ въ правой части всѣ три члена дисъюнктны другъ съ другомъ. А потому

$$N(a+b)=N(ab)+N(ab_0)+N(a_0b).$$

Сравненіе этого выраженія съ найденнымъ выше показываетъ намъ, что вообще

$$N(a+b)=N(a)+N(b)-N(ab),$$

откуда, въ частности, для случая, когда $ab=0$ и слѣд. $N(ab)=N(0)=0$, получимъ, какъ и ранѣе,

$$N(a+b)=N(a)+N(b).$$

Далѣе, легко видѣть, что вообще (въ силу доказаннаго, а также закона $mm=m$):

$$N(a+b+c)=N[(a+b)+c]=N(a+b)+N(c)-N[(a+b)c]=$$
$$=N(a)+N(b)-N(ab)+N(c)-N[ac+bc]=$$
$$=N(a)+N(b)+N(c)-N(ab)-[N(ac)+N(bc)-N(abc)]=$$
$$=[N(a)+N(b)+N(c)]-[N(ab)+N(ac)+N(bc)]+N(abc).$$

Точно такъже мы нашли-бы:

$$N(a+b+c+\partial)=[N(a)+N(b)+N(c)+N(\partial)]-$$
$$-[N(ab)+N(ac)+N(a\partial)+N(bc)+N(b\partial)+N(c\partial)]+$$
$$+[N(abc)+N(ab\partial)+N(bc\partial)]-$$
$$-N(abc\partial).$$

Законъ построенія этихъ формулъ очевиденъ. Въ частности, когда всѣ слагаемые классы дисъюнктны между собою, мы находимъ:

$$N\Sigma a^{(i)}=\Sigma N(a^{(i)}),$$

откуда, по раздѣленіи на $N(1)$, получаемъ отношеніе:

$$P(a'+a''+a'''+\ldots)=P(a')+P(a'')+P(a''')+\ldots.,$$

т. е. еще одну истину Теоріи Вѣроятностей, на которую мы ссылались выше.

Можно найти иное выраженіе для символа $N\Sigma a^{(i)}$.

Такъ какъ въ логикѣ имѣетъ мѣсто тождество:

$$a'+a''=a'+a'_0a'',$$

гдѣ въ правой части оба члена дисъюнктны между собою, то

$$N(a'+a'')=N(a')+N(a'_0a'').$$

Далѣе, зная, что

$$a'+a''+a'''=a'+a'_0a''+a'_0a''_0a''',$$

гдѣ опять всѣ члены правой части дисьюнктны между собою, найдемъ:

$$N(a'+a''+a''')=N(a')+N(a'_0a'')+N(a'_0a''_0a''').$$

Точно такъ-же найдемъ и вообще:

$$N(a'+a''+a'''+\ldots)=N(a')+N(a'_0a'')+N(a'_0a''_0a''')+\ldots$$

Третій пріемъ для опредѣленія символа $N\Sigma a^{(i)}$ заключается въ разложеніи суммы $\Sigma a^{(i)}$ на элементы объема (которые всегда дисъюнктны между собою). Поэтому, если такое разложеніе есть:

$$\Sigma a^{(i)}=s'+s''+s'''+\ldots,$$

то понятно, что

$$N\Sigma a^{(i)}=N\Sigma s^{(k)}.$$

Наконецъ, четвертый пріемъ опредѣленія того-же символа есть слѣдующій. Такъ какъ отрицаніе суммы $a'+a''+$ $+a'''+\ldots$ есть произведеніе $a'_0a''_0a'''_0\ldots$, то понятно, что

$$N(a'+a''+a'''+\ldots)=N(1)-N(a'_0a''_0a'''_0\ldots).$$

Обращаемся къ опредѣленію символа N отъ произведенія логическихъ классовъ.

Выше было доказано, что

$$N(a+b)=N(a)+N(b)-N(ab),$$

а потому

$$N(ab) = N(a) + N(b) — N(a+b).$$

Легко также видѣть, что

$$N(ab) = N(1) — N[(ab)_0] = N(1) — N(a_0 + b_0) \quad . \quad . \quad . \quad . \quad (E)$$

Обобщеніемъ этихъ формулъ я заниматься не буду. Вмѣсто того, обращу вниманіе на слѣдующее. Предпослѣдняя формула показываетъ намъ, что, зная символы $N(a)$ и $N(b)$, мы еще не можемъ опредѣлить величины символа $N(ab)$. Однако, легко указать предѣлы, внутри которыхъ содержится величина этого символа; а именно: $N(ab)$ не меньше нуля и не больше наименьшаго изъ символовъ $N(a)$ и $N(b)$.

Буль доказалъ, что нижній изъ этихъ предѣловъ можно формулировать обстоятельнѣе. А именно, онъ доказываетъ, что $N(ab)$ не меньше

$$N(a) + N(b) — N(1).$$

Въ самомъ дѣлѣ, изъ формулы (E) слѣдуетъ, что

$$\begin{aligned} N(ab) &= N(1) - N(a_0 + b_0) = N(1) — [N(a_0) + N(b_0) — N(a_0 b_0)] = \\ &= N(1) — [N(1) — N(a) + N(1) - N(b) — N(a_0 b_0)] = \\ &= N(a) + N(b) - N(1) + N(a_0 b_0). \end{aligned}$$

Вотъ новое выраженіе для символа $N(ab)$, откуда видимъ, что, дѣйствительно, $N(ab)$ не меньше

$$N(a) + N(b) - N(1).$$

Далѣе, для случая трехъ множителей имѣемъ:

$$N(a'a''a''')=N[(a'a'')a''']=N(a'a'')+N(a''')-N(1)+$$
$$+N((a'_0+a''_0)a'''_0)=$$
$$=N(a')+N(a'')-N(1)+N(a'_0a''_0)+N(a''')-N(1)+$$
$$+N((a'_0+a''_0)a'''_0)=N(a')+N(a'')+N(a''')-2N(1)+$$
$$+[N(a'_0a''_0)+N((a'_0+a''_0)a'''_0)].$$

Такъ какъ каждый изъ символовъ N не меньше нуля, то отсюда слѣдуетъ, что $N(a'a''a''')$ не меньше

$$N(a')+N(a'')+N(a''')-2N(1).$$

Такими-же сужденіями можно доказать, что вообще

$$N(a'a''a'''...a^{(m)}) \text{ не меньше } \Sigma N(a)-(m-1)N(1).$$

Таковъ нижній передѣлъ величины символа N отъ произведенія классовъ. Что-же касается верхняго, то понятно, что величина того-же символа не больше величины наименьшаго изъ символовъ $N(a'), N(a''),..., N(a^{(m)})$.

Вотъ собственно и все, что мнѣ извѣстно о правилахъ нумеризаціи.

Въ заключеніе замѣчу слѣдующее. Выше мы получили изъ правилъ нумеризаціи двѣ основныя истины Теоріи Вѣроятностей. Однако, дальнѣйшія истины той-же науки мы можемъ получить изъ правилъ нумеризаціи только при помощи гипотезы о равномѣрномъ распредѣленіи предметовъ каждаго класса по всему протяженію міра рѣчи. Напримѣръ, только при условіи этой гипотезы, мы можемъ сказать, что $N(ab)$ составляетъ ту-же часть отъ $N(a)$, какъ $N(b)$ отъ $N(1)$, т. е. написать пропорцію.

$$N(ab):N(a)=N(b):N(1),$$

откуда

$$N(ab) = \frac{N(a)N(b)}{N(1)},$$

и слѣдов., по раздѣленіи на $N(1)$:

$$P(ab) = \frac{N(a)}{N(1)} \cdot \frac{N(b)}{N(1)} = P(a)P(b).$$

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Президентъ Общества *А. Штукенбергъ.*

Казань. Типографія Университета 1887 года.

Biography of P.S. Poreckii

Figure 1: Platon Sergeevič Poreckii.

The photo obtained by the courtesy of Ms. Galina Alexandrovna Aukhadieva, the Director of the N.I. Lobachevsky Scientific Library of the Kasan State University, Kasan, Russia.

A detailed biography of Poreckij can be found in [3] and [8]. We present here basic facts related to his professional work.

P.S. Poreckij received his High School education at the Gymnasium in Poltava, and then continued at the Faculty of Physics and Mathematics, the University of Kharkov, Ukraine, where he graduated in 1870. In the period 1871 - 1874, he worked at the same University as a student of the Professor of Astronomy I.I. Fedorenko. After that, he moved to the Astronomy Observatory in Astrahan, Poulkova, St. Petersburg, and finally in 1876 to the University of Kasan.

Poreckij received his Master degree in Astronomy in May 25 1886, at the Faculty for Physics and Mathematics of the Kasan University. The master thesis was very positively reviewed and highly recommended by the famous astronomer Professor D.I. Dubjago, the Director of the University Astronomy Observatory in Kasan, due to which the Council of the Kasan University decided to award Poreckij with the degree of Doctor of Astronomy on a meeting held on May 31, 1889. The diploma was written already on March 12, 1889, and was presented to Poreckij on April 5, 1889. For a period of time, Poreckij served as the secretary of the Kasan Society of Natural Sciences. Because of weak health, Poreckij retired from the Kasan University on his own request submitted to the Rector of the Kasan University on March 4, 1889, however, preserving his cooperation within the Kasan Society of Natural Sciences.

Scientific Work of Poreckij

As it was pointed out in [4], Platon Sergeevič Poreckij was motivated to study logic by the famous mathematician A.V. Vasilev, and father of the founder of imaginary logic N.A. Vasliev, see also [13].

As documented in [2], [3], [4], Platon Sergeevich Poreckij was the first who gave a course in Mathematical Logic in Russia, at the University of Kasan.

Poreckij has been primarily interested in logic equations and inequalities, and application of mathematical logic in probability theory. As noticed in [3], the method developed by Poreckij in this area has been more universal than approaches by Jevons and Venn [15], [16], at least as it has been estimated by Couturat [7].

In [12], it is given a first attempt at a complete theory of qualitative inference, where under the term *quality* Poreckij meant *one-place predicate* in modern terminology [1], see also [10].

A detailed information about Poreckij, and his work including also a biography can be found in [2], [3], [4], [8], and [14].

Publications of P.S. Poreckij

The following list (in Russian) of publications by P.L. Poreckij has been compiled by V.A. Bazhanov [3], [4]. We added item 20, the review of which appeared in [5]. In the same paper, the items 2, 3, 7, 13, 15, 17, 18, 19, 22, 23, and 24 have been reviewed. Item 20 has been also referred in [6], together with the items 17, 18, and 19.

1. *Determining of geographic latitude of the Astronomical Tower of the University of Kharkov*, Kharkov, Ukraine, 1873, 57 pages.

2. "Presentation of fundamental principles of mathematical logic in more evident and popular form", Presentation at the Third meeting of the Kasan Society for Natural Sciences, *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 1, 1881, 2-31.

3. "On methods for solving logical equations and the inverse method of mathematical logic", *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 2, 1884, No. XXIV, separate publication, 170 pages.

4. "On the question of solving certain normal systems appearing in spherics astronomy, with applications in determination of the errors in the division of the meridian circle of the Observatory of Kasan", (four presentations in 1885), Kasan, Russia, 1886, separate publication, 144 pages.

5. "On the relationship between the days in a year and the days in a week", *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 4, 1886, separate publication, 12 pages.

6. "Historical notice on the development of spherics trigonometry", *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 5, 1887, separate publication, 16 pages.

7. "Solving general tasks in Probability Theory by using Mathematical Logic", *Collection of Records of Meetings of the Section for Physic and*

Mathematics of the Scientific Society of the Kasan University, Kasan, Russia, Vol. 5, 1887, 83-116, (Translated into German).

8. "Mars-Opposition im Jahre 1879", (P. Poretzki) *Astronomische Nachrichten*, Vol. 116, No. 24, 1887, 369-372.

9. "Four observations, Mars opposition 1877, Mars opposition 1879, Mars opposition 1886, Beobachtungen des Cometen 1881 III", (P. Poretzki) *Astronomische Nachrichten* Vol. 117, No. 8, 1887, 131-132.

10. "Determination of the geographic latitude of the Astronomical Tower of the Kharkov University", *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 6, 1888, separate publication, 58 pages.

11. "Apropos of the presentation by P.V. Preobrazhensky *Trigonometric series of a particular form*, Presentations at The 76th Meeting of the Section of Physics and Mathematics of the Scientific Society of the Kasan University, *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 7, 1888, 330-334.

12. "Apropos of the publication by Mr. Cesarski *Astronimic photometry*", *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 7, 1888, 334-339.

13. "Apropos of the brochure by Mr. Volkov *Logic Calculations*", Presentation on November 12, at the 81st Meeting of the Section for Physics and Mathematics of the Scientific Society of the Kasan University, *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, Vol. 7, 1888, separate publication, 9 pages.

14. "On the theory of prime numbers", *Collection of Records of Meetings of the Section for Physic and Mathematics of the Scientific Society of the Kasan University*, Kasan, Russia, 1888, Vol. 6.

15. "La loi de racines en logique (The law of roots in Logic)", *Revie de mathématiques (revista di matematica)*, 1896-9, Vol. 6, No. 19, 538-593.

16. ”New science and the Academician Imšeneckij (with attachment of three letters by Imšeneckij)”, *Nordic Newsletter*, December 1896, 103-112.

17. ”Sept lois fondamenteles de la théorie de égalités logiques”, *Bulletin of the Society for Physics and Mathematics of the Kazan University*, Second series, Vol. 8, 1899, 33-103, 129-181, 183-216. Also a separate publication *Sept lois fondamenteles de la théorie de égalités logiques*, Kasan, Typo-lithography of the Imperial University, 1899, Vol. II, 157 pages.

18. ”Exposé élementaire de la théorie des égalités logiques à deux termes *a* and *b*”, *Revue de métaphysique et de morale*, 1900, Vol. 8, 169-188.

19. ”Quelques lois ultéuieures de la théorie des égalités logiques (Supplément au traité - Sept lois foundamentales de la théorie des égalités logiques ”, *Bulletin of the Society for Physics and Mathematics of the Kazan University*, Second series, Vol. 10, No. 1, 1900, 50-84, No. 2, 1900, 132-180, No. 3, 1990, 191-230, Vol. 11, No. 2, 1901, 17-63. There is a separate publication *Quelques lois ultéuieures de la théorie des égalités logiques*, Kazan, Typo-lithography of the Imperial University, 1902, No. V, 163 pages.

20. ”Théorie des égalités logiques à trois termes, *a*, *b*, et *c*”, *Bibliothèque du Congrès International de Philosophie*, Paris, Vol. 3, 1901, 201-233.

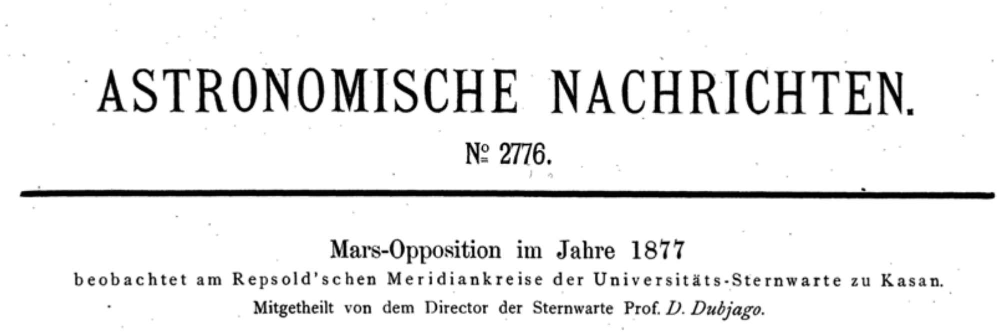
21. ”From the area of mathematical logic”, *Physic-Mathematic Year-book Devoted to the Questions of Mathematics, Physics, Chemistry, Astronomy and Elementary Presentations*, Second Year, Moscow, Publishing house of the group of authors *Proceedings in the Aid of Self-education*, 1902, No. 2, 482 pages.

22. ”Théorie des non-égalités logiques. Supplément aux deux traités - *Sept lois fondamentales de la théorie des égalités logiques* et *Quelques lois ultérieures de la théorie des égalités logiques*”, *Bulletin of the Society for Physics and Mathematics of the Kasan University*, Second series, 1904, Vol. 13, No. 3, 80-119, No. 4, 1904, 127-184. There is a separate edition *Theorie des non-egalites logiques*, Kasan, Typo-lithography of the Imperial University, 1904, Vol. III, 112 pages.

23. "Appendice sur mon nouvel travail "Théorie des non-égalités logiques", *Bulletin of the Society for Physics and Mathematics of the Kasan University*, Second series, Vol. 14, No. 2, 1904, 118-131.

24. "Theorie conjointe des égalités et des non-égalités logiques", *Bulletin of the Society for Physics and Mathematics of the Kasan University*, Second series, Vol. 16, No. 1-2, 1908, 9-41 and Vol. 16, No. 2, 1910, 41-118. There is a separate edition *Theorie conjointe des égalités et des non-égalités logiques*, Kasan, Typo-lithography of the Imperial University, 1909, Vol. III, 109 pages.

Fig. 2 shows the title of the paper reported above as the item 9, and Fig. 3 shows the closing paragraph of this article where it is written the name of Poreckij as the person who did the observations.



ASTRONOMISCHE NACHRICHTEN.

№ 2776.

Mars-Opposition im Jahre 1877

beobachtet am Repsold'schen Meridiankreise der Universitäts-Sternwarte zu Kasan.

Mitgetheilt von dem Director der Sternwarte Prof. *D. Dubjago.*

Figure 2: The title of the paper reported as the item 9 in the bibliography of Poreckij.

Stern	α app.	B—R	δ app.	B—R
W. 1261	23^h 1^m $2^s.08$		−12° 27′ 54″.6	
W. 57 (med.)	23 5 37.62		−12 35 39.5	
Lal. 45504	23 9 0.06		−12 13 42.7	
ψ^3 Aquarii	23 12 37.83		−10 16 36.6	
W. 315	23 16 58.56		−10 3 10.4	
W. 402	23 21 45.21		−12 7 10.6	
W. 497	23 26 7.58		−11 40 17.8	
W. 571	23 29 9.77		−11 13 41.9	
W. 629	23 31 55.39		− 9 18 4.5	
1877 Oct. 21.				
58 Aquarii	22 25 13.88		−11 31 46.5	
64 Aquarii	22 32 51.67		−10 39 42.9	
74 Aquarii	22 47 4.05		−12 15 52.1	
Mars	22 51 19.69	$-0^s.71$	−10 26 57.8	−1″.2
W. 1156	22 56 9.64		−11 55 13.1	
W. 1261	23 1 1.99		−12 27 54.1	
W. 57 (med.)	23 5 37.43		−12 35 39.5	
Lal. 45504	23 8 59.90		−12 13 44.2	
ψ^3 Aquarii	23 12 37.78		−10 16 36.8	
W. 315	23 16 58.50		−10 3 10.8	
W. 402	23 21 45.14		−12 7 11.7	

Stern	α app.	B—R	δ app.	B—R
W. 497	23^h 26^m $7^s.56$		−11° 40′ 19″.1	
W. 571	23 29 9.76		−11 13 43.7	
B.A.C. 8239	23 34 51.10		−12 21 22.0	
B.A.C. 8285	23 43 57.96		−10 39 16.0	
1877 Oct. 23.				
58 Aquarii	22 25 13.96		−11 31 48.7	
64 Aquarii	22 32 51.70		−10 39 43.1	
74 Aquarii	22 47 3.96		−12 15 51.4	
Mars	22 52 42.39	$-0^s.34$	−10 9 22.3	−1″.9
W. 1156	22 56 9.68		−11 55 12.6	
W. 1261	23 1 1.99		−12 27 53.4	
W. 57 (med.)	23 5 37.37		−12 35 40.9	
Lal. 45504	23 8 59.97		−12 13 43.5	
ψ^3 Aquarii	23 12 37.73		−10 16 35.5	
W. 315	23 16 58.47		−10 3 12.4	
W. 402	23 21 45.12		−12 7 11.7	
W. 497	23 26 7.40		−11 40 18.4	
W. 571	23 29 9.74		−11 13 44.2	
B.A.C. 8239	23 34 50.97		−12 21 21.4	
B.A.C. 8285	23 43 58.03		−10 39 16.9	

Die ersten 5 Beobachtungen sind von Prof. M. Kowalski, die übrigen von dem Observator P. Poretzki angestellt worden.

Aug. 18, 20 und 25 Bilder unruhig; Aug. 26, 27, Sept. 1, 4 und 18 Mars in Wolken.

Sept. 4 und 8 ist statt des südlichen Randes der nördliche, Oct. 21 und 23 statt des nördlichen das Centrum des Mars eingestellt.

Mittlere Oerter der Vergleichsterne für 1877.0.

Stern	α 1877.0	Autorität	δ 1877.0	Autorität
58 Aquarii	22^h 25^m $9^s.88$	1/3 (Pulk.+Cap+Gl.)	−11° 32′ 6″.8	1/4 (Pulk.+Cap+Gl.+Y.)
64 Aquarii	22 32 47.62	1/2 (Pulk.+Cap)	−10 40 2.4	1/2 (Pulk.+Cap)
94 Aquarii	23 12 38.40	1/3 (Pulk.+Cap+Sj.)	−14 7 40.1	1/3 (Pulk.+Cap+Sj.)
B.A.C. 8239	23 34 46.82	1/4 (Pulk.+Cap+Gl.+Sj.)	−12 21 44.8	1/4 (Pulk.+Cap+Gl.+Sj.)
B.A.C. 8285	23 43 53.75	1/4 (Cap+Gl.+Y.+Sj.)	−10 39 39.2	1/4 (Cap+Gl.+Y.+Sj.)

Der Ort von λ Aquarii ist dem Fundamental-Catalog von Auwers, die Oerter der übrigen Vergleichsterne sind der Gill'schen Bestimmung (Monthly Not. 39) entnommen.

Die sämmtlichen Vergleichungen beziehen sich auf die Transit-Ephemeride des Nautical Almanac.

Kasan 1887 Februar 16.

Figure 3: The closing paragraph of the paper reported as the item 9 in the bibliography of Poreckij.

References about the Work of Poreckij

The biography and work by Poreckij has been explored in detail and reported by V.A. Bazhanov in [2], [3], [4]. The famous astronomer D. Dubjago, a contemporary of Poreckij, was a reviewer of the Master thesis of Poreckij and a friend of him reporting his work [8].

In [7], the work by Poreckij in logic equalities is presented as an original method more universal than the corresponding works by S. Jevons and J. Venn [9], [16].

The work by Poreckij has been mentioned in the review by George L. Kline of the work by S.A. Yanovskaya (Janovskaja) [1], [10], and in the review by Werner Stelzner [13] of the book by V.A. Bazhanov [2].

In [11], it is written at page 2,
There was in prerevolutionary Russia a certain tradition of studies in symbolic logic, going back to Poreckij, who published his first work as early as 1881 - two years after the "Begriffsschrift" [3].

Fig. 4 shows a part of the text of the review by Kline discussing the work by Poreckij as it was reported by Yanovskaya. This text appears at the page 46 in [10].

Turning to the historical development of mathematical logic, the author mentions the work of De Morgan, Boole, Jevons, Peirce, and Schröder, and goes on to state that "the culmination of this period . . . was the work of the Russian logician, P. S. Poréckij, Lobačévskij's colleague at Kazan University" (p. 19). Poréckij considered his 582 the first attempt at a complete theory of qualitative inference (by a "quality" Poréckij meant what is now called a "one-place predicate").

Figure 4: A part of the text at page 46 in [10].

[3]The book by Gottlob Frege published in 1879, viewed by some scholars as the most important publication in logic after Aristotle founded the subject. The complete reference of the book is *Begriffsschrift, eine der Arithmetischen Nachgebildete Formelsprache des Reinen Denkens*, Halle A/8, Verlag von Louis Nebert, 1879.

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